Quantum Expanders and Geometry of Operator Spaces

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Abstract

We show that there are well separated families of quantum expanders with asymptotically the maximal cardinality allowed by a known upper bound. This has applications to the "local theory" of operator spaces. This allows us to provide sharp estimates for the growth of the multiplicity of M_N-spaces needed to represent (up to a constant $C > 1$) the M_N-version of the n-dimensional operator Hilbert space OH_n as a direct sum of copies of M_N. We show that, when C is close to 1, this multiplicity grows as $\exp \beta n N^2$ for some constant $\beta > 0$. The main idea is to identify quantum expanders with "smooth" points on the matricial analogue of the unit sphere. This generalizes to operator spaces a classical geometric result on n-dimensional Hilbert space (corresponding to $N = 1$). Our work strongly suggests to further study a certain class of operator spaces that we call matricially subGaussian.

In a second part, we introduce and study a generalization of the notion of exact operator space that we call subexponential. Using Random Matrices we show that the factorization results of Grothendieck type that are known in the exact case all extend to the subexponential case, and we exhibit (a continuum of distinct) examples of non-exact subexponential operator spaces. We also show that OH, $R + C$ and $\max(\ell_2)$ (or any other maximal operator space) are not subexponential.

The term "Quantum Expander" is used by Hastings in [14] to designate a sequence $\{U^{(N)} \mid N \geq 1\}$ of n-tuples $U^{(N)} = (U_1^{(N)}, \cdots, U_n^{(N)})$ of $N \times N$ unitary matrices such that there is an $\varepsilon > 0$ satisfying the following "spectral gap" condition:

$$(0.1) \qquad \forall N\ \forall x \in M_N \quad \|\sum\nolimits_1^n U_j^{(N)}(x - tr(x)){U_j^{(N)}}^* \|_2 \leq n(1-\varepsilon)\|x - tr(x)\|_2,$$

where $\|.\|_2$ denotes the Hilbert-Schmidt norm on M_N. More generally, the term is extended to the case when this is only defined for infinitely many N's, and also to n-tuples of matrices satisfying merely $\sum U_j^{(N)}{U_j^{(N)}}^* = \sum {U_j^{(N)}}^* U_j^{(N)} = nI$.
We will say that an n-tuple $U^{(N)}$ satisfying (0.1) is a ε-quantum expander.

In analogy with the classical expanders (see below), one seeks to exhibit (and hopefully to construct explicitly) sequences $\{U^{(N_m)} \mid m \geq 1\}$ of n-tuples of $N_m \times N_m$ unitary matrices that are ε-quantum expanders with $N_m \to \infty$ while n and $\varepsilon > 0$ remain fixed.

When G is a finite group generated by $S = \{t_1, \cdots, t_n\}$ the associated Cayley graph $\mathcal{G}(G, S)$ is said to have a spectral gap if the regular representation λ_G satisfies

$$\|\sum \lambda_G(t_j)_{|\mathbb{I}^\perp}\| < n(1-\varepsilon) \tag{0.2}$$

where $\mathbb{I}$ denotes the constant function 1 on G. Obviously, this is equivalent to the condition that the unitaries $U_j = \lambda_G(t_j)$ satisfy (0.1) when restricted to diagonal matrices x (here $N = |G|$). In this light, quantum expanders appear as a non-commutative version of the classical ones.
More precisely, (0.2) holds iff the unitaries $U_j = \lambda_G(t_j)$ satisfy (0.1) for all x in the orthogonal complement of right translation operators. This is easy to deduce from the decomposition into irreducibles of $\lambda_G \otimes \bar{\lambda}_G$, in which the component of the trivial representation corresponds to the restriction to right translation operators.

A sequence of Cayley graphs $\mathcal{G}(G^{(m)}, S^{(m)})$ constitutes an expander in the usual sense if (0.2) is satisfied with $\varepsilon > 0$ and n fixed while $|G^{(m)}| \to \infty$.

Expanders (equivalently expanding graphs) have been extremely useful, especially (in the applied direction) since Margulis and Lubotzky-Phillips-Sarnak obtained explicit constructions (as opposed to random ones). We refer to [20, 15] for more information and references.
They have also been used with great success for operator algebras and in operator theory (see e.g. [39, 5, 16] see also [30, 2]). In [16], is crucially used the fact that when the dimensions N, N' are suitably different, say if N is much larger than N', and $U^{(N)}$ satisfies (0.1) then $U^{(N)}$ and $U^{(N')}$ are separated in the sense that there is a fixed $\delta = \delta(\varepsilon) > 0$ such that $\forall x \in M_{N \times N'} \quad \|\sum U_j^{(N)} x U_j^{(N')^*}\|_2 \le n(1-\delta)\|x\|_2$ (see Remark 1.13 for more on this).

Motivated by operator theory considerations, it is natural to wonder what happens when $N = N'$. We will say that two n-tuples $u = (u_j)$ and $v = (v_j)$ of $N \times N$ unitary matrices are δ-separated if

$$\forall x \in M_N \quad \|\sum\nolimits_1^n u_j x {v_j}^*\|_2 \le n(1-\delta)\|x\|_2.$$

Equivalently this means that

$$\|\sum\nolimits_1^n u_j \otimes \bar{v_j}\| \le n(1-\delta)$$

where $\bar{v_j}$ denotes the complex conjugate of the matrix v_j, and the norm is the operator norm on $\ell_2^N \otimes \overline{\ell_2^N}$. This can be interpreted in operator space theory as a rough sort of orthogonality related to the "operator space Hilbert space OH".

Note for example that when (0.2) holds then, for any pair of inequivalent irreducible representations π, σ on G, the n-tuples $(\pi(t_j))$ and $(\sigma(t_j))$ are ε-separated.

Let $U(N) \subset M_N$ denote the group of unitary matrices. The main result of §1 asserts that there exists absolute constants $\beta > 0$ and $\delta > 0$ such that for each $0 < \varepsilon < 1$, for all sufficiently large integers n and N, there is a δ-separated family $\{u(t) \mid t \in T\} \subset U(N)^n$ of n-tuples of ε-quantum expanders such that

$$|T| \ge \exp \beta n N^2.$$

Thus we can "pack" as many as $m = \exp \beta n N^2$ δ-separated ε-quantum expanders inside $U(N)^n$. This number m is remarkably large. In fact, in some sense it is the largest possible. Indeed, it is known ([6], see also Remark 1.7) that the maximal m is at most $\exp \beta' n N^2$ for some absolute constant β'.

In §2, we use quantum expanders to prove the analogue for operator spaces of a well known geometric property of Euclidean space: The unit sphere in a Hilbert space is smooth. Equivalently all its points admit a unique norming functional. In our extension of this, "norming" will be with respect to the operator space duality. Moreover, unicity has to be understood modulo an

equivalence relation: for any $x = (x_j) \in M_N(E)^n$ we define $Orb(x)$ as the set of all x' of the form $x' = (ux_jv) \in M_N(E)^n$ for some $u, v \in U(N)$. Then if x is "norming" some point, any $x' \in Orb(x)$ is also 'norming" that same point. When E is an operator space and $x \in M_N(E)$, we will say that $y \in M_N(E^*)$ M_N-norms x if $\|\sum x_j \otimes y_j\| = \|x\|_{M_N(E)} \|y\|_{M_N(E^*)}$. We will say that x is M_N-smooth in $M_N(E)$ if the only points y with $\|y\|_{M_N(E^*)} = 1$ that M_N-norm x are all in a single orbit in $M_N(E^*)$. Let us now turn to the case $E = OH_n$. Then we show that, if $x \in U(N)^n$ is viewed as an element of $M_N(\ell_2^n)$, then x is M_N-smooth in $M_N(E)$ iff x is an ε-quantum expander for some $\varepsilon > 0$.

More generally, in Lemma 1.12 we prove a more precise quantified version of this: if x is an ε-quantum expander and if two points $y, z \in M_N(E^*)$ both M_N-norm x up to some error δ, then the distance of the orbits $Orb(y)$ and $Orb(z)$ is uniformly small, i.e. majorized a function $f(\varepsilon, \delta)$ that tends to 0 when $(\varepsilon, \delta) \to (0, 0)$. Here the distance is meant with respect to the renormalized Euclidean norm $y \mapsto (nN)^{-1/2}\|y\|_2$ for which any $y \in U(N)^n$ has norm 1 (where $\|.\|_2$ denotes here the norm in $\ell_2(n \times N^2)$).

This also has a geometric application. Consider the following problem for an n-dimensional normed space E: Given a constant $C > 1$, estimate the minimal number $k = k_E(C)$ of functionals $f_1, \cdots f_k$ in the dual E^* such that

$$\forall x \in E \quad \sup_{1 \le j \le k} |f_j(x)| \le \|x\| \le C \sup_{1 \le j \le k} |f_j(x)|.$$

Geometrically this means that (in the real case) the symmetric convex body that is the unit ball of E^* is equivalent (up to the factor C) to a polyhedron with vertices included in $\{\pm f_j\}$ and hence with at most $2k$ vertices (so its polar, that is equivalent to the unit ball of E, has at most $2k$ faces). For instance, the n-dimensional cube has 2^n vertices and $2n$ faces. When E has (real) dimension n it is well known (see e.g. [28, p.49-50]) that

$$k_E(C) \le (\frac{3C}{C-1})^n.$$

For example if $C = 2$ we have $k_E(C) \le 6^n$. This exponential order of growth in n is optimal for $E = \ell_2^n$ (or ℓ_p^n for $1 \le p < \infty$) but of course $k_E(C) = n$ for $E = \ell_\infty^n$, and there is important available information and a conjecture (see [25]) about conditions on a general sequence $\{E(n) \mid n \ge 1\}$ with $\dim(E(n)) = n$ ensuring that $k_{E(n)} \ge \exp cn$ for some $c > 0$.

We now describe the matricial analogue of k_E that we estimate using quantum expanders. Let E be an operator space. Fix an integer $N \ge 1$. We denote by $k_E(N, C)$ the smallest k such that there are linear maps $f_j : E \to M_N$ $(1 \le j \le k)$ satisfying

$$\forall x \in M_N(E) \quad \sup_{1 \le j \le k} \|(Id \otimes f_j)(x)\|_{M_N(M_N)} \le \|x\|_{M_N(E)} \le C \sup_{1 \le j \le k} \|(Id \otimes f_j)(x)\|_{M_N(M_N)}.$$

It is not hard to adapt the corresponding Banach space argument to show that for *any* n-dimensional E, any $C > 1$ and any N we have

$$k_E(N, C) \le (\frac{3C}{C-1})^{2nN^2}.$$

Using the "packing" of ε-quantum expanders described above, we can show that the operator space version of Hilbert space (i.e. the space OH from [29]) satisfies a lower bound of the same order of growth, namely we show for $E = OH_n$ (see Theorem 2.8) there are numbers $C_1 > 1$, $b > 0$ such that for any n, N large enough we have

$$k_{OH_n}(N, C_1) \ge \exp bnN^2.$$

This suggests that the class of finite dimensional operator spaces E such that $\log k_{OH_n}(N,C)/N^2 \to 0$ should be investigated. We call such spaces matricially subGaussian.

In [16, 35] operator space versions of Grothendieck's theorem were proved in the form of a special factorization property for (jointly) completely bounded bilinear forms on $E \times F$ when A, B are C^*-algebras and $E \subset A$, $F \subset B$ are *exact* operator subspaces. In particular, when $E = A, F = B$ this was proved for exact C^*-algebras. In [13] this last result was extended to arbitrary C^*-algebras. A remarkable, considerably simpler proof was recently given in [36]. In the case of "exact" subspaces $E \subset A, F \subset B$, the proof in [36] deduces the result of [35] directly from that of [16]. In this paper we introduce a larger class of operator spaces, that we call "subexponential", for which the same Grothendieck type factorization property from [16, 35] still holds. The known examples of non-exact operator spaces turn out to be also non-subexponential, but in §6 an example is constructed showing that the new class is strictly larger than that of exact operator spaces. The definition of "subexponential" involves the growth of a sequence of integers $N \mapsto K_E(N,C)$ attached to an operator space E (and a constant $C > 1$), in a way that is similar but seems different from $k_E(N,C)$. We denote by $K_E(N,C)$ the smallest K such that there is a single (embedding) linear map $f: \ E \to M_K$ satisfying

$$\forall x \in M_N(E) \quad \|(Id \otimes f)(x)\|_{M_N(M_K)} \le \|x\|_{M_N(E)} \le C\|(Id \otimes f)(x)\|_{M_N(M_N)}.$$

Roughly the latter sequence is bounded iff E is exact while it is such that $\log K_E(N,C)/N \to 0$ iff E is subexponential. For the non-exact example constructed in §6, we even have polynomial growth: we have $K_E(N, 2+\varepsilon) \in O(N^2)$. There is a notion of "subexponential constant" analogous to the exactness constant, and we give estimates from below (of the same order) of that constant for the same examples (OH_n, $R_n + C_n$ or maximal spaces) for which lower bounds of the exactness constant are known.

To tackle subexponentiality, we make crucial use of Gaussian random matrices and particularly of [11]. Let $Y^{(N)}$ denote a random $N \times N$-matrix with i.i.d. complex Gaussian entries with mean zero and second moment equal to $N^{-1/2}$, and let $(Y_j^{(N)})$ be a sequence of i.i.d. copies of $Y^{(N)}$. Let $E \subset B(H)$ be an operator space. For any $(a_1, \cdots, a_n) \in E^n$ we define

$$|||(a_1, \cdots, a_n)||| = \limsup_{N\to\infty} \left\| \sum_1^n a_j \otimes Y_j^{(N)} \right\|, \tag{0.3}$$

where-here and below-the norm is the minimal (or spatial) tensor norm (here this is simply the norm of $M_N(B(H))$. Note that this is non-random. Indeed, by concentration of measure (see e.g. [18]) we have almost surely

$$\limsup_{N\to\infty} \left| \left\| \sum_1^n a_j \otimes Y_j^{(N)} \right\| - \mathbb{E} \left\| \sum_1^n a_j \otimes Y_j^{(N)} \right\| \right| = 0,$$

and hence

$$|||(a_1, \cdots, a_n)||| = \limsup_{N\to\infty} \mathbb{E} \left\| \sum_1^n a_j \otimes Y_j^{(N)} \right\|. \tag{0.4}$$

The main result of [12] implies that if E is 1-exact then

$$|||(a_1, \cdots, a_n)||| = \| \sum a_j \otimes C_j \| \tag{0.5}$$

where (C_j) is a free circular sequence in Voiculescu's sense, and the limsup is actually almost surely a limit. Indeed, it can be shown rather easily (by weak convergence) that

$$\|\sum a_j \otimes C_j\| \le \liminf_{N\to\infty} \left\|\sum_1^n a_j \otimes Y_j^{(N)}\right\|$$

holds almost surely for *any* E.
This result (0.5) was recently extended to unitary matrices (and a few other cases) by Collins and Male, see [4]. In that case, $Y^{(N)}$ is uniformly distributed over the unitary group $U(N)$ of all $N \times N$ unitary matrices, and (C_j) has to be replaced by a free family of Haar unitaries.

If E is C-exact, then [12] implies $|||(a_1, \cdots, a_n)||| \le C\|\sum a_j \otimes C_j\|$. A fortiori this implies the following result proved in [11] (prior to [12]): If E is C-exact, then

(0.6) $\quad \forall n\ \forall (a_1, \cdots, a_n) \in E^n \quad |||(a_1, \cdots, a_n)||| \le 2C \max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}.$

A somewhat simpler proof can be derived from [3], see [34, Cor. 15.3] for details.

The starting point of our study of subexponential spaces is the observation (that we made several years ago after reading [11]) that (0.6) remains valid if E is C-subexponential. It might be worthwhile to study the class of operator spaces satisfying (0.6) for some C. Moreover, it would be interesting to understand what kind of operator spaces appear in (0.4) in the non subexponential case.
Note: There is an obvious upper bound (for a fixed constant C) $K_E(N, C) \le N k_E(N, C)$, so the growth of K_E is dominated by that of k_E, but we know nothing in the converse direction. Various other questions are mentioned at the end of §3.

1. Quantum Expanders

Fix integers n, N. Throughout this paper we denote by M_N the space of $N \times N$ complex matrices and by $U(N)$ the subset of $N \times N$ unitary matrices.
We identify M_N with the space $B(\ell_2^N)$ of bounded operators on the N-dimensional Hilbert space denoted by ℓ_2^N.

We denote by tr (resp. τ_N) the usual trace (resp. the normalized trace) on M_N. Thus $\tau_N = N^{-1}\mathrm{tr}$. We denote by S_2^N the Hilbert space obtained by equipping M_N with the corresponding scalar product. The associated norm is the classical Hilbert-Schmidt norm.

For simplicity we denote by

$$H = L_2(\tau_N),$$

i.e. H is the Hilbert space obtained by equipping the space M_N with the norm

$$\|\xi\|_H = (N^{-1}\mathrm{tr}(|\xi|^2)^{1/2} = N^{-1/2}\|x\|_{S_2^N}.$$

We denote

$$H_0 = \{I\}^\perp \subset H.$$

Throughout this paper, we consider operators of the form $T = \sum x_j \otimes \bar{y}_j$, with $x_j, y_j \in M_N$, that we view as acting on $\ell_2^N \otimes \overline{\ell_2^N}$. Identifying as usual $\ell_2^N \otimes \overline{\ell_2^N}$ with S_2^N, we may consider T as an operator acting on M_N defined by

$$\forall \xi \in M_N \quad T(\xi) = \sum x_j \xi {y_j}^*,$$

and we then have $\|\sum x_j \otimes \bar{y}_j : \ \ell_2^N \otimes \overline{\ell_2^N} \to \ell_2^N \otimes \overline{\ell_2^N}\| = \|T : \ S_2^N \to S_2^N\|$. Actually it will be convenient to view T as an operator acting on $H = L_2(\tau_N)$. We have trivially

$$\|T\|_{B(H)} = \|T\|_{B(S_2^N)}.$$

Let $x = (x_j) \in (M_N)^n$ and $y = (y_j) \in (M_N)^n$. Let $Orb(x)$ denote the 2-sided unitary orbit of $x = (x_j)$, i.e.

$$Orb(x) = \{(ux_jv) \mid u, v \in U(N)\}.$$

We will denote

$$d(x, y) = (\sum\nolimits_j \|x_j - y_j\|^2_{L_2(\tau_N)})^{1/2},$$

and

$$d'(x, y) = \inf\{d(x', y) \mid x' \in Orb(x)\} = \inf\{d(x', y') \mid x' \in Orb(x), y' \in Orb(y)\}.$$

The last equality holds because of the 2-sided unitary invariance of the norm in S_2^N or equivalently of $H = L_2(\tau_N)$.

Definition 1.1. Fix $\delta > 0$. We will say that x, y in M_N^n are δ-separated if

$$\|\sum x_j \otimes \bar{y}_j\| \le (1 - \delta)\|\sum x_j \otimes \bar{x}_j\|^{1/2}\|\sum y_j \otimes \bar{y}_j\|^{1/2}.$$

A family of elements is called δ-separated if any two distinct members in it are δ-separated.

Let $x = (x_j) \in M_N^n$ and $y = (y_j) \in M_N^n$ be normalized so that $\|\sum x_j \otimes \bar{x}_j\| = \|\sum y_j \otimes \bar{y}_j\| = 1$. Equivalently, this definition means that for any $\xi, \eta \in M_N$ in the unit ball of S_2^N we have

$$|\sum \mathrm{tr}(x_j \xi y_j^* \eta^*)| \le 1 - \delta.$$

Using polar decompositions $\xi = u|\xi|$ and $\eta = v|\eta|$, $|\sum \mathrm{tr}(x_j \xi y_j^* \eta^*)| = |\sum \mathrm{tr}(x_j u|\xi| y_j^* |\eta| v^*)|$. Let $\hat{x}_j = v^* x_j u$. Equivalently we have for any u, v unitary

$$|\sum \mathrm{tr}(\hat{x}_j |\xi| y_j^* |\eta|)| \le 1 - \delta.$$

A fortiori, taking $|\xi| = |\eta| = N^{-1/2} I$ we find $|\sum \tau_N(\hat{x}_j y_j^*)| \le 1 - \delta$ and hence

$$d(\hat{x}, y)^2 \ge 2\delta$$

and hence taking the inf over u, v unitary, the δ-separation of x, y implies

$$d'(x, y) \ge (2\delta)^{1/2}. \tag{1.1}$$

In other words, rescaling this to the case when $n^{1/2}x_j, n^{1/2}y_j, \xi, \eta$ are all unitary, we have proved:

Lemma 1.2. *Consider n-tuples $x = (x_j) \in U(N)^n$ and $y = (y_j) \in U(N)^n$. If x, y are δ-separated then $d'(x, y) \ge (2\delta n)^{1/2}$.*

Recall that we denote

$$H_0 = \{I\}^\perp.$$

To any n-tuple $u = (u_j) \in U(N)^n$ we associate the operator $(\sum u_j \otimes \bar{u}_j)(1 - P)$ on $\ell_2^N \otimes \overline{\ell_2^N}$ where P denotes the $\perp$-projection onto the orthogonal of $I = \sum e_j \otimes \bar{e}_j$. Equivalently, up to the normalization, we will consider

$$T : \ H_0 \to H_0$$

defined for all $\xi \in H_0$ by

$$T(\xi) = \sum u_j \xi u_j^*.$$

We will denote by $S_\varepsilon = S_\varepsilon(n, N) \subset U(N)^n$ the set of all n-tuples $u = (u_j) \in U(N)^n$ such that

$$\|T: \; H_0 \to H_0\| \le \varepsilon n.$$

Equivalently, this means $\forall x \in M_N$, we have

$$\| \sum u_j (x - \tau_N(x) I) u_j^* \|_H \le \varepsilon n \|x\|_H.$$

Definition 1.3. Let $0 < \varepsilon < 1$. An n-tuple $u = (u_j) \in U(N)^n$ will be called ε-Ramanujan if

$$\|T: \; H_0 \to H_0\| \le 2\sqrt{n-1} + \varepsilon n.$$

We will denote by $R_\varepsilon = R_\varepsilon(n, N) \subset U(N)^n$ the set of all such n-tuples

We refer to [20, 15] for more information on expanders and Ramanujan graphs.

Remark 1.4. Recall (see [30, p. 324. Th. 20.1]) that for any n-tuple of unitary operators on any Hilbert space H we have

$$\| \sum u_j \otimes \bar{u}_j \| \ge 2\sqrt{n-1}.$$

Note that $2\sqrt{n-1} < n$ for all $n \ge 3$ (so there is also an $0 < \varepsilon < 1$ such that $2\sqrt{n-1} + \varepsilon n < n$). Our goal is to prove the following:

Theorem 1.5. *There are absolute constants $\beta > 0$ and $\delta > 0$ such that for each $0 < \varepsilon < 1$ and for all sufficiently large integers n and N, more precisely such that $n \ge n_0$ and $N \ge N_0$ with n_0 depending on ε, and N_0 depending on n and ε, there is a δ-separated subset*

$$T \subset R_\varepsilon$$

such that

$$|T| \ge \exp \beta n N^2.$$

Remark 1.6. Actually, the proof will show that if we are given sets $A_N \subset U(N)^n$ such that $\alpha = \inf_N \mathbb{P}(A_N) > 0$, then we can find a subset T as above with $T \subset A_N \cap R_\varepsilon$, but with δ, β, n_0 and N_0 now also depending on α.

Remark 1.7. The order of growth of our lower bound $\exp \beta n N^2$ in Theorem1.5 is roughly optimal because of the upper bound given in [6]. The latter upper bound can be proved as follows. Let $m_{\max}$ be the maximal number of a δ-separated family in $U(N)^n$. Consider the normed space obtained by equipping $M(N)^n$ with the norm $|||x||| = \| \sum x_j \otimes \bar{x}_j \|^{1/2}$. Then since its (real) dimension is $2nN^2$, by a well known volume argument ([28, p.49-50]) there cannot exist more than $(1 + 2/\delta')^{2nN^2}$ elements in its unit ball at mutual $|||.|||$-distance $\ge \delta'$. Note that $d(x, y) \le |||x - y|||$ for any pair x, y in $M(N)^n$. Thus, if $u, v \in U(N)^n$ are δ-separated in the above sense then $x = n^{-1/2} u$ and $y = n^{-1/2} v$ are in the $|||.|||$-unit ball and by (1.1) we have $|||x - y||| \ge (2\delta)^{1/2}$, therefore

$$m_{\max} \le (1 + \sqrt{2/\delta})^{2nN^2} \le \exp\{2\sqrt{2/\delta} \; nN^2\}.$$

Remark 1.8. Let G be a Kazhdan group (see [1]) with generators $t_1, \cdots, t_n$, so that there is $\delta > 0$ such that $\|\sum_1^n \pi(t_j)\| \le n(1-\delta)$ for any unitary representation without any invariant (non zero) vector. Let $\mathcal{I} = \mathcal{I}(N)$ denote the set of N-dimensional irreducible representations $\pi: G \to U(N)$. It is known (see [1]) that the latter set is finite and in fact there is a uniform bound on $|\mathcal{I}(N)|$ for each N. For any $\pi \in \mathcal{I}$ we set

$$u_j^\pi = \pi(t_j).$$

Then

$$\sup_{\pi \neq \sigma \in \mathcal{I}} \|\sum u_j^\pi \otimes \overline{u_j^\sigma}\| \le n(1-\delta),$$

so that the family $\{u^\pi \mid \pi \in \mathcal{I}\} \subset U(N)^n$ is δ-separated in the above sense. By the preceding Remark, we know $|\mathcal{I}(N)| \le m_{\max} \le \exp c_\delta n N^2$. The problem to estimate the maximal possible value of $|\mathcal{I}(N)|$ when $N \to \infty$ (with δ and n remaining fixed, but G possibly varying) is investigated in [22]: some special cases are constructed in [22] for which $|\mathcal{I}(N)|$ grows like $\exp cN$, however we feel that Theorem 1.5 gives evidence that there should exist cases for which $|\mathcal{I}(N)|$ grows like $\exp cN^2$.

We will make crucial use of a result due to Hastings [14]:

Lemma 1.9 (Hastings)**.** *If we equip $U(N)^n$ with its normalized Haar measure $\mathbb{P}$, then for each n and $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}(R_\varepsilon(n, N)) = 1.$$

We will also use the following Lemma that is inspired by a non-commutative variant of results in [27] (see also [26] where the non-commutative case is already considered) in the style of [21] (see also [8, 24]). We view this as a (weak) sort of non-commutative Sauer lemma, that it might be worthwhile to strengthen.

Lemma 1.10. *Let $a > 0$. Let $A \subset U(N)^n$ be a (measurable) subset with $\mathbb{P}(A) > a$. Then, for any $c < \sqrt{2}$, A contains a finite subset $T \subset A$ with*

$$|T| \ge a \exp KrnN^2,$$

such that

$$\forall s \neq t \in T \quad d(s,t) \ge c\sqrt{n}$$

where $r = (1 - c^2/2)^2$ and K is a universal constant.
Assuming moreover that $a \ge \exp -KnN^2/8$, we find that A contains a finite subset $T \subset A$ with

$$|T| \ge \exp bnN^2,$$

such that

$$\forall s \neq t \in T \quad d(s,t) \ge \sqrt{n}$$

where $b = K/8$ is an absolute constant.

Proof. Let $\Omega = U(N)^n$. We may clearly assume (by Haar measure inner regularity) that A is compact. Let $T \subset A$ be a maximal finite subset such that $\forall s \neq t \in T$ $d(s,t) \ge c\sqrt{n}$. Then, by the maximality of T, A is included in the union of the open balls with center $t \in T$ and d-radius $c\sqrt{n}$. By translation invariance of d and $\mathbb{P}$, all these balls have the same $\mathbb{P}$-measure equal to $F(c)$. Therefore $a < \mathbb{P}(A) \le |T|F(c)$ and hence

$$aF(c)^{-1} < |T|.$$

Thus we need a lower bound for $F(c)^{-1}$. Let u denote the unit in $U(N)^n$ so that $u_j = 1$ for $1 \le j \le n$. Using a ball centered at u to compute $F(c)$, we have

$$F(c) = \mathbb{P}\{\omega \in U(N)^n \mid \sum\nolimits_1^n \mathrm{tr}(|\omega_j - 1|^2) < c^2 nN\}.$$

Since $\sum_1^n \mathrm{tr}(|\omega_j - 1|^2) = 2Nn - 2\sum_1^n \Re\mathrm{tr}(\omega_j)$, we have

$$F(c) = \mathbb{P}\{\omega \mid \sum\nolimits_1^n \Re\mathrm{tr}(\omega_j) > nN(1 - c^2/2)\}.$$

We will now use the known subGaussian property of $\sum_1^n \Re\mathrm{tr}(\omega_j)$: there is a universal constant K such that for any $\lambda > 0$ we have

$$\mathbb{P}\{\omega \mid \sum\nolimits_1^n \Re\mathrm{tr}(\omega_j) > \lambda\} \le \exp -K\lambda^2/n. \tag{1.2}$$

Taking this for granted, let us complete the proof. Fix $c < \sqrt{2}$. Recall $r = (1 - c^2/2)^2 > 0$, this yields

$$F(c) \le \exp -KnN^2 r.$$

Thus we conclude that

$$|T| > a \exp KrnN^2.$$

In particular taking $c = 1, r = 1/4$, we see that if we assume $a \ge \exp -KnN^2/8$ we find $|T| > \exp(KnN^2/8)$.

Let us now give a quick argument for the known inequality (1.2): We will denote by $Y^{(N)}$ a random $N \times N$-matrix with i.i.d. complex Gaussian entries with mean zero and second moment equal to $N^{-1/2}$, and we denote by $(Y_j^{(N)})$ a sequence of i.i.d. copies of $Y^{(N)}$. It is well known that the polar decomposition $Y^{(N)} = U|Y^{(N)}|$ is such that U is uniformly distributed over $U(N)$ and independent of $|Y^{(N)}|$. Moreover if we let $\chi_N > 0$ be defined by $\mathbb{E}|Y^{(N)}| = \chi_N I$, then we have $\chi = \inf_N \chi_N > 0$. See e.g. [21, p. 80]. Therefore, we have a conditional expectation operator $\mathcal{E}$ (corresponding to integrating the modular part) such that $\sum \Re\mathrm{tr}(\omega_j) = \chi_N^{-1}\mathcal{E}(\sum \Re\mathrm{tr}(Y_j^{(N)})$, where ω_j denotes the unitary part in the polar decomposition of $Y_j^{(N)}$.
Then, since $x \mapsto \exp wx$ is convex for any $w > 0$, we have the announced subGaussian property

$$\mathbb{E}\exp w \sum \Re\mathrm{tr}(\omega_j) \le \mathbb{E}\exp w\chi_N^{-1} \sum \Re\mathrm{tr}(Y_j^{(N)}) = \exp(\chi_N^{-2} w^2 n/4),$$

from which follows, by Markov's inequality, that $\mathbb{P}\{\sum \Re\mathrm{tr}(\omega_j) > \lambda\} \le \exp(\chi^{-2}w^2 n/4 - \lambda w)$ and optimising w so that $\lambda = \chi^{-2}wn/2$ we finally obtain

$$\mathbb{P}\{\sum \Re\mathrm{tr}(\omega_j) > \lambda\} \le \exp(-K\lambda^2/n),$$

with $K = \chi^2$. The above simple argument follows [21, ch. 5], but, in essence, (1.2) can traced back to [9, Lemma 3]. □

The next Lemma is a simple covering argument.

Lemma 1.11. *Fix $b, c > 0$. Let $T \subset U(N)^n$ be a subset with $|T| \ge \exp bnN^2$ and such that $d(s,t) \ge c\sqrt{n}\ \forall s \ne t \in T$. Fix $c' < c/2$ and $b' < b$. Then there is an integer n_0 such that if $n \ge n_0$ there is a subset $T' \subset T$ with $|T'| \ge \exp b'nN^2$ and such that $d'(s,t) \ge c'\sqrt{n}\ \forall s \ne t \in T'$.*

Proof. Let $T' \subset T$ be a maximal subset such that $d'(s,t) \geq c'\sqrt{n}\ \forall s \neq t \in T'$. Then for any $t \in T$ there is $t'(t) \in T'$ such that $d'(t,t'(t)) < c'\sqrt{n}$. This means that there are $u(t), v(t) \in U(N)$ such that $d(t, u(t)t'(t)v(t)) < c'\sqrt{n}$. Fix $\varepsilon > 0$. It is well known (this is an easy case of [37][p. 175]) that there is an ε-net $S \subset U(N)$ with respect to the operator norm with $|S| \leq (K/\varepsilon)^{2N^2}$ (indeed we can even cover the unit ball of M_N and argue by a simple volume argument such as in [28][p. 49-50]). In any case, this gives us $|S| \leq \exp 2N^2 \log(K/\varepsilon)$.
Replacing $u(t), v(t)$ by their approximation in S, we can find $\hat{u}(t), \hat{v}(t) \in S$ such that $\|u(t) - \hat{u}(t)\| \leq \varepsilon$ and $\|v(t) - \hat{v}(t)\| \leq \varepsilon$. By an easy argument this yields $d(t, \hat{u}(t)t'(t)\hat{v}(t)) < c'\sqrt{n} + 2\varepsilon\sqrt{n}$.
Assume that $|T'| < \exp b'nN^2$. Then

$$|S \times T' \times S| < \exp(N^2(b'n + 4\log(K/\varepsilon))).$$

There is clearly an integer $n_0(\varepsilon)$ such that $b'n + 4\log(K/\varepsilon) < bn$ for all $n \geq n_0(\varepsilon)$, so that $|S \times T' \times S| < |T|$. Then by the pigeon hole principle, there must exist $t_1 \neq t_2$ in T such that $\hat{u}(t_1)t'(t_1)\hat{v}(t_1) = \hat{u}(t_2)t'(t_2)\hat{v}(t_2)$. Let us denote by θ this common value. We have then $d(t_1, \theta) < (c' + 2\varepsilon)\sqrt{n}$ and $d(t_2, \theta) < (c' + 2\varepsilon)\sqrt{n}$. Therefore $d(t_1, t_2) < 2(c' + 2\varepsilon)\sqrt{n}$. Now if $c' < c/2$ we can choose $\varepsilon > 0$ such that $2(c' + 2\varepsilon) < c$. By our assumption on T, this is impossible. This contradiction completes the proof. □

Lemma 1.12. *Fix $0 < \varepsilon < 1$. Consider $u = (u_j) \in S_\varepsilon \subset U(N)^n$. Consider the function $\delta \mapsto f(\delta)$ defined by $f(\delta) = \sqrt{2}(\delta + 2\sqrt{2}(2\delta - \delta^2 + \varepsilon)^{1/2})^{1/2}$. (For small δ this is $\approx 2^{5/4}(2\delta + \varepsilon)^{1/4}$).*
Then for any $0 < \delta < 1$ and any $v = (v_j) \in M_N^n$ such that $\|\sum v_j \otimes \bar{v}_j\| \leq n$ the condition

$$d'(u,v) \geq f(\delta)\sqrt{n} \tag{1.3}$$

implies

$$\|\sum u_j \otimes \bar{v}_j\| \leq n(1-\delta). \tag{1.4}$$

Conversely, it is easy to show that for any $v = (v_j) \in U(N)^n$ (or merely such that $\sum \tau_N(|v_j|^2) = n$)

$$\|\sum u_j \otimes \bar{v}_j\| \leq n(1-\delta),$$

implies

$$d'(u,v) \geq \sqrt{2n\delta}.$$

Proof. Assume by contradiction that $\|\sum u_j \otimes \bar{v}_j\| > n(1-\delta)$. Then there are ξ, η in the unit sphere of $H = L_2(\tau_N)$ such that

$$\Re\tau_N(\sum u_j \xi v_j^* \eta^*) > n(1-\delta).$$

Let $\xi = U|\xi|$ and $\eta = V|\eta|$ be their polar decompositions, and let $w_j = V^* u_j U$ so that we can write

$$\Re\tau_N(\sum w_j |\xi| v_j^* |\eta|) > n(1-\delta). \tag{1.5}$$

Note that since $u \otimes \bar{u}$ and $v \otimes \bar{v}$ preserve I (and hence $I^\perp$), we have $\|T_w\| = \|T_u\|$. Therefore $w \in S_\varepsilon$. Using the scalar product in H we have

$$\sum \langle |\eta|^{1/2} w_j |\xi|^{1/2}, |\eta|^{1/2} v_j |\xi|^{1/2} \rangle > n(1-\delta),$$

and by Cauchy-Schwarz (here the norm is in H)

$$(\sum \||\eta|^{1/2} w_j |\xi|^{1/2}\|^2)^{1/2}\ (\sum \||\eta|^{1/2} v_j |\xi|^{1/2}\|^2)^{1/2} > n(1-\delta).$$

Note that since $\|\sum v_j \otimes \bar{v}_j\| \leq n$ we have $\langle (\sum v_j \otimes \bar{v}_j)|\xi|, |\eta|\rangle = \sum \||\eta|^{1/2} v_j |\xi|^{1/2}\|^2 \leq n$, and similarly with w_j in place of v_j. Thus the last inequality implies a fortiori

$$\langle (\sum w_j \otimes \bar{w}_j)|\xi|, |\eta|\rangle > n(1-\delta)^2, \tag{1.6}$$

and the same with v_j in place of w_j. Let $e = (1-P)|\xi|$ and $d = (1-P)|\eta|$. Note

$$\langle (\sum w_j \otimes \bar{w}_j)|\xi|, |\eta|\rangle = \langle T_w e, d\rangle + n\tau_N(|\xi|)\tau_N(|\eta|) \leq \varepsilon n \|e\|\|d\| + n\tau_N(|\xi|)\tau_N(|\eta|).$$

Since $\tau_N(|\xi|), \tau_N(|\eta|)$ are both ≤ 1, (1.6) yields both

$$\|P|\xi|\| = \tau_N(|\xi|) > (1-\delta)^2 - \varepsilon \quad \text{and} \quad \|P|\eta|\| = \tau_N(|\eta|) > (1-\delta)^2 - \varepsilon$$

and hence assuming $(1-\delta)^2 - \varepsilon > 0$

$$\||\xi| - 1\|^2 = 2 - 2\tau_N(|\xi|) < 2(1-(1-\delta)^2+\varepsilon) \quad \text{and} \quad \||\eta| - 1\|^2 = 2 - 2\tau_N(|\eta|) < 2(1-(1-\delta)^2+\varepsilon).$$

Let $\theta = (2(1-(1-\delta)^2+\varepsilon))^{1/2}$, so that $\||\xi|-1\| \leq \theta$ and $\||\eta|-1\| \leq \theta$. This implies

$$\Re\tau_N(\sum w_j|\xi|v_j^*|\eta|) \leq \Re\tau_N(\sum w_j v_j^*) + \Re\tau_N(\sum w_j v_j^*(|\eta|-1)) + \Re\tau_N(\sum w_j(|\xi|-1)v_j^*|\eta|)$$

$$\leq \Re\tau_N(\sum w_j v_j^*) + 2\theta n.$$

Going back to (1.5) we find

$$\Re\tau_N(\sum w_j v_j^*) > n(1-\delta-2\theta).$$

The latter is the real part of the scalar product in $\ell_2^n(H)$ of $w = (w_j)$ and $v = (v_j)$ which are both in the ball of radius $\sqrt{n}$. Therefore we deduce from this

$$d(w,v)^2 < 2n - 2\Re\tau_N(\sum w_j v_j^*) < 2n(\delta + 2\theta).$$

Note that $f(\delta) = (2(\delta+2\theta))^{1/2}$, so that we have $d(w,v) < f(\delta)\sqrt{n}$.
Thus we have proved that $\|\sum u_j \otimes \bar{v}_j\| > n(1-\delta)$ implies $d'(u,v) < f(\delta)\sqrt{n}$, at least provided $(1-\delta)^2 - \varepsilon > 0$. This is equivalent to the fact that (1.3) implies (1.4). Moreover, if $(1-\delta)^2 - \varepsilon \leq 0$ then $f(\delta) > 2$ so that (1.3) is impossible.

Now assume conversely that (1.4) holds. Note that (1.4) remains true for any $w \in Orb(u)$: Indeed, for all $s, t \in U(N)$, we have $\sum su_jt \otimes \bar{v}_j = (s \otimes I)(\sum u_j \otimes \bar{v}_j)(t \otimes I)$ and hence $\|\sum su_jt \otimes \bar{v}_j\| = \|\sum u_j \otimes \bar{v}_j\|$. Therefore it suffices to show that (1.4) implies $d(u,v) \geq \sqrt{2\delta n}$. Then, using $\xi = \eta = I$, we find

$$\Re\tau_N(\sum u_j v_j^*) \leq (1-\delta)n.$$

Since we assume u_j, v_j all unitary, we have $\sum \|v_j\|_H^2 = n = \sum \|u_j\|_H^2$. Therefore,

$$d(u,v)^2 = \sum \|u_j - v_j\|_H^2 = 2n - 2\Re\tau_N(\sum u_j v_j^*) \geq 2\delta n.$$

□

Proof of Theorem 1.5. Fix $\varepsilon > 0$. When n is large enough (say $n \geq n_0(\varepsilon)$) we have clearly $R_{\varepsilon/2} \subset S_\varepsilon$, and hence by Hastings' Lemma 1.9 there is $N_0(\varepsilon)$ such that $\mathbb{P}(S_\varepsilon) > 1/2$ with $a = 1/2$ for all $N \geq N_0(\varepsilon)$. By Lemmas 1.10 and 1.11, assuming n large enough (say $n \geq n_0$) there is $T' \subset S_\varepsilon$ with $|T'| \geq \exp b'nN^2$ such that $d'(s,t) \geq c'\sqrt{n}$ for all $s \neq t \in T'$, where b', c' are positive absolute

constants. We now invoke Lemma 1.12. Note that $\lim_{\delta\to 0} f(\delta) = 2^{5/4}\varepsilon^{1/4}$. Therefore for any $\varepsilon \le (c'/8)^4$ we find $\lim_{\delta\to 0} f(\delta) \le c'/2$. Therefore there is a numerical value of $\delta > 0$ such that $f(\delta) < c'$, and hence $d'(s,t) \ge f(\delta)\sqrt{n}$ for all $s \ne t \in T'$. Then Lemma 1.12 ensures that the set T' is δ-separated for that value of δ. Since we have $T' \subset S_\varepsilon \subset R_\varepsilon$ we obtain the desired conclusion (when $\varepsilon \le (c'/8)^4$) with $\beta = b'$. This completes the proof since (by the monotony of $\varepsilon \to R_\varepsilon$) the remaining case $\varepsilon > (c'/8)^4$ is trivially a consequence of the one when $\varepsilon \le (c'/8)^4$. □

Remark 1.13. Lemma 1.12 has the following consequence: Assume $u = (u_j) \in S_\varepsilon$ and let $f(\delta)$ be as in Lemma 1.12. Then for any $v = (v_j) \in U(k)^n$ with $k \le (1 - f(\delta)^2)N$ we have

$$\|\sum u_j \otimes \bar{v}_j\| \le n(1-\delta).$$

Indeed, if $\|\sum u_j \otimes \bar{v}_j\| > n(1-\delta)$, then we set $v'_j = v_j \oplus 0 \in M_N$ so that $\|\sum v'_j \otimes \bar{v}'_j\| \le n$, and also $\|\sum u_j \otimes \bar{v}'_j\| > n(1-\delta)$. By Lemma 1.12, it follows that $d'(u,v') < f(\delta)\sqrt{n}$. But since $|\langle u'_j, v'_j\rangle| \le k/N$ for any $u'_j \in U(N)$, we have $d(u',v')^2 = n + n(k/N) - 2\sum\langle u'_j, v'_j\rangle \ge n(1-k/N)$, and hence $d'(u,v') \ge \sqrt{n}(1-k/N)^{1/2}$, which leads to $(1-k/N)^{1/2} < f(\delta)$. This contradiction concludes the proof.

2. Application to Operator Spaces

We start with a specific notation. Let $u:\ E \to F$ be a linear map between operator spaces. We denote for any given $N \ge 1$

$$u_N = Id \otimes u:\ M_N(E) \to M_N(F).$$

Moreover, if E, F are two operator spaces that are isomorphic as Banach spaces, we set

$$d_N(E,F) = \inf\{\|u_N\|\|(u^{-1})_N\|\}$$

where the inf runs over all the isomorphisms $u:\ E \to F$. We set $d_N(E,F) = \infty$ if E, F are not isomorphic.
Recall that

$$\|u\|_{cb} = \sup_{N\ge 1} \|u_N\|.$$

Recall also that, if E, F are completely isomorphic, we set

$$d_{cb}(E,F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb}\}$$

where the inf runs over all the complete isomorphisms $u:\ E \to F$.

We will apply the preceding to M_N-spaces. When $N = 1$, the latter coincide with the usual Banach spaces. When $N > 1$, roughly the complex scalars are replaced by M_N.

Let $(A_i)_{i\in I}$ be a family of von Neumann or C^*-algebras. Let $Y = \oplus_{i\in I} A_i$ denote their direct sum. This can be described as the algebra of bounded families $(a_i)_{i\in I}$ with $a_i \in A_i$ for all $i \in I$, equipped with the norm $\|a\| = \sup_{i\in I}\|a_i\|$. We will concentrate on the case when $A_i = M_N$ for all $i \in I$. In that case, following the Banach space tradition, we denote the space $Y = \oplus_{i\in I} A_i$ by $\ell_\infty(I; M_N)$.

Definition 2.1. An operator space X is called an M_N-space if, for some set I, it can be embedded completely isometrically in $\ell_\infty(I; M_N)$.

Our main interest will be to try to understand for which spaces the cardinality of I is unusually small.

To place things in perspective, we recall that for any (complex) Banach space X there is an isometric embedding $J: X \to \ell_\infty(I;\mathbb{C})$ defined by $(Jx)(\phi) = \phi(x)$. Here I is the unit ball, denoted by B_{X^*}, of the space X^*.

In analogy with this, for any M_N-space there is a canonical completely isometric embedding $\hat{J}: X \to \ell_\infty(\hat{I}; M_N)$ defined again by $(Jx)(\phi) = \phi(x)$, but with $\hat{I} = B_{CB(X,M_N)}$ in place of B_{X^*}. The space $\ell_\infty(\hat{I}; M_N)$ can alternatively be described as $\oplus_{i\in\hat{I}} Z_i$ with $Z_i = M_N$ for all $i \in \hat{I}$.

Just like operator spaces, M_N-spaces enjoy a nice duality theory (see [19, 23] for more information). Indeed, by Roger Smith's lemma, we have $\|u\|_{cb} = \|u_N\|$ for any u with values in an M_N-space (see e.g. [30, p. 26]), and M_N-spaces are characterized among operator spaces by this property. The following reformulation of Smith's Lemma is useful.

Lemma 2.2. *Fix an integer $N \geq 1$. Let $E \subset B(H)$ be a finite dimensional operator space and let $c \geq 1$ be a constant. The following properties are equivalent.*
(i) *For any operator space F and any $u: F \to E$ we have $\|u\|_{cb} \leq c\|u_N\|$.*
(ii) *There is an M_N-space such that $d_{cb}(E, \hat{E}) \leq c$.*
(iii) *Let $\mathcal{C}$ be the class of all (compression) mappings $v: E \to B(H', H'')$ of the form $x \mapsto P_{H''} x_{|H'}$ where H', H'' are arbitrary subspaces of H of dimension at most N. Let $\hat{J}: E \to \oplus_{v\in\mathcal{C}} Z_v$ with $Z_v = B(H', H'')$ be defined by $\hat{J}(x) = \oplus_{v\in\mathcal{C}} v(x)$, and let $\hat{E} = \hat{J}(E)$. Then $d_{cb}(E, \hat{E}) \leq c$.*

Proof. (ii) $\Rightarrow$ (i) follows from Roger Smith's lemma and (iii) $\Rightarrow$ (ii) is trivial. Conversely, if (i) holds, let $\hat{E}$ be the M_N-space obtained using the embedding $\hat{J}: E \to \oplus_{v\in\mathcal{C}} Z_v$ appearing in (iii). Obviously $\|E \to \hat{E}\|_{cb} \leq 1$. Let us denote by $u: \hat{E} \to E$ the inverse mapping. A simple verification shows that $\|u_N\| = 1$ and hence (i) implies $\|u\|_{cb} \leq c$. In other words (i) $\Rightarrow$ (iii). $\square$

Therefore, when X is an M_N-space, the knowledge of the space $M_N(X)$ determines that of $M_n(X)$ for all $n > N$, and hence the whole operator space structure of X.

Given a general operator space $X \subset B(H)$, by restricting to $M_N(X)$ (and "forgetting" $M_n(X)$ for $n > N$), we obtain an M_N-space M_N-isometric to X. We will say that the latter M_N-space is induced by X.
Conversely, given an M_N-space X there is a minimal and a maximal operator space structure on X inducing the same M_N-space. When $N = 1$, we recover the Blecher-Paulsen theory of minimal and maximal operator spaces associated to Banach spaces, see [19, 23] for more on this.

Let E be a finite dimensional operator space. For each integer N, let $E[N]$ denote the induced M_N-space. Then it is easy to check that E can be identified (completely isometrically) with the ultraproduct of $\{E[N]\}$ relative to any free ultraproduct on $\mathbb{N}$. Thus the operator space structure of E can be encoded by the sequence of M_N-spaces $\{E[N] \mid N \geq 1\}$. Note that $E[N]$ is induced by $E[N+1]$ for any N, so that one could picture the set of n-dimensional operator spaces as infinite branches of trees where the N-th node consists of an M_N-space, and any node is induced by any successor.

We can associate to each M_N-space a dual one $X^\dagger$, isometric to the operator space dual X^*, but defined by

$$\forall n \in \mathbb{N} \quad \forall y \in M_n(X^\dagger) \quad \|y\|_{M_n(X^\dagger)} = \sup\{\|(I \otimes f)(y)\|_{M_n(M_N)} \mid f \in M_N(X), \|f\|_{M_N(X)} \leq 1\},$$

where we view $M_N(X)$ as a subset of $CB(X^*, M_N)$ in the usual way. In other words we have a completely isometric embedding $J_\dagger: X^\dagger \to \ell_\infty(I; M_N)$ defined by

$$J_\dagger(z) = \oplus_{f\in B_{M_N(X)}} f(z) = \oplus_{f\in B_{M_N(X)}} [f_{ij}(z)].$$

Just like for operator spaces, there is a notion of "Hilbert space" for M_N-spaces. We will denote it by $OH(n, N)$. The latter can be defined as follows. Fix N. Let $S(n, N)$ (resp. $B(n, N)$) denote the set of n-tuples $x = (x_j)$ in M_N such that $\|\sum x_j \otimes \bar{x_j}\| = 1$ (resp. $\|\sum x_j \otimes \bar{x_j}\| \leq 1$). Then $S(n, N)$ (resp. $B(n, N)$) is the analogue of the unit sphere (resp. ball) in the M_N-space $OH(n, N)$. The space $X = OH(n, N)$ is isometric to ℓ_2^n, with its orthonormal basis (e_j), and embedded into $\ell_\infty(I; M_N)$ with $I = B(n, N)$ (we could also take $I = S(n, N)$). The embedding $J_{oh} : OH(n, N) \to \ell_\infty(I; M_N)$ is defined by

$$\forall j = 1, \cdots, n \quad J_{oh}(e_j) = \oplus_{x \in B(n,N)} x_j.$$

The latter is the analogue of n-dimensional Hilbert space among M_N-spaces, and indeed when $N = 1$ we recover the n-dimensional Hilbert space.

Definition 2.3. Let E be an operator space with basis (e_j). Let ξ_j be the biorthogonal basis of E^*. Let $x = \sum x_j \otimes e_j \in M_N(E)$ and $y = \sum y_j \otimes \xi_j \in M_N(E^*)$. Assuming $x \neq 0$ and $y \neq 0$, we say that y M_N-norms x (with respect to $M_N(E)$) if

$$\|\sum x_j \otimes y_j\| = \|x\|_{M_N(E)} \|y\|_{M_N(E^*)}.$$

In the particular case when $E = OH_n$, we slightly modify this (since $E^* = \bar{E}$): Given $x, y \in M_N(OH_n)$, we say that y M_N-norms x if

$$\|\sum x_j \otimes \bar{y}_j\| = \|\sum x_j \otimes \bar{x}_j\|^{1/2} \|\sum y_j \otimes \bar{y}_j\|^{1/2}.$$

Let $x \in M_N(E)$. For $a, b \in M_N$ we denote by axb the matrix product (i.e. $(a \otimes 1)x(b \otimes 1)$ in tensor product notation using $M_N(E) = M_N \otimes E$). We denote

$$Orb(x) = \{uxv \in M_N(E) \mid u, v \in U(N)\}.$$

Note that if $y \in M_N(E^*)$ M_N-norms x then the same is true for any $y' \in Orb(y) \subset M_N(E^*)$. Actually, any $y' \in Orb(y)$ M_N-norms any $x' \in Orb(x)$.

Definition 2.4. We say that $x \in M_N(E)$ is an M_N-smooth point of $M_N(E)$ if the set of points y in the unit sphere of $M_N(E^*)$ that M_N-norm x is reduced to a single orbit.

The following simple Proposition explains the direction we will be taking next.

Proposition 2.5. *Let $x, y \in M_N(OH_n)$. Assume*

$$x = (x_j) \in U(N)^n \text{ and } \|T_x : \ H_0 \to H_0\| < n,$$

where $T_x = \sum x_j \otimes \bar{x}_j(1 - P)$ (i.e. T_x has a spectral gap at n).
Then y norms x with respect to $M_N(OH_n)$ iff y is a multiple of an element of $Orb(x)$, i.e. iff there are $\lambda > 0$ and $u, v \in U(N)$ such that $y_j = \lambda v x_j u$ for all $1 \leq j \leq n$.

Proof. Recall that whenever the x_j's are finite dimensional unitaries we have $\|\sum x_j \otimes \bar{x}_j\|^{1/2} = \sqrt{n}$. Assume y is a multiple of an element of $Orb(x)$, i.e. $y_j = \lambda v x_j u$ for some non zero scalar λ (that may as well be taken positive if we wish). Then $\|\sum x_j \otimes y_j\| = |\lambda| n$, $\|x\|_{M_N(OH_n)} = \sqrt{n}$ and $\|y\|_{M_N(OH_n)} = |\lambda|\sqrt{n}$, so indeed y norms x.
Conversely, assume that y norms x. Multiplying y by a scalar we may assume that $\|y\|_{M_N(OH_n)} =$

$\sqrt{n}$, and $\|\sum x_j \otimes \bar{y}_j\| = \|\sum x_j \otimes \bar{x}_j\|^{1/2}\sqrt{n} = n$. Let ξ, η in the unit sphere of $H = L_2(\tau_n)$ such that

$$\sum \tau_N(x_j \xi y_j^* \eta^*) = n.$$

Let $\xi = u|\xi|$ and $\eta = v|\eta|$ be the polar decompositions, and let $x'_j = v^* x_j u$. Using the trace property, this can be rewritten using the scalar product in H as:

$$\sum \langle (|\eta|^{1/2} x'_j |\xi|^{1/2}), (|\eta|^{1/2} y_j |\xi|^{1/2}) \rangle = n,$$

and hence since $n^{-1/2}(|\eta|^{1/2} x'_j |\xi|^{1/2}), n^{-1/2}(|\eta|^{1/2} y_j |\xi|^{1/2})$ are both in the unit ball of the (smooth!) Hilbert space $\ell_2^n(H)$, they must coincide. Moreover they both must be on the unit sphere. Therefore $\sum \||\eta|^{1/2} x'_j |\xi|^{1/2}\|_H^2 = n$. Equivalently $\sum \tau_N(x'_j |\xi| x'^*_j |\eta|) = n$. But we have obviously $\|T_{x'} : H_0 \to H_0\| = \|T_x : H_0 \to H_0\| < n$. Therefore $|\xi|$ and $|\eta|$ must be multiples of I, so that by our normalization we have $|\xi| = |\eta| = I$, and we conclude that $y = x'$. □

In other words, the preceding Proposition shows that quantum expanders constitute M_N-smooth points of $M_N(OH_n)$:

Corollary 2.6. *Assume $x = (x_j) \in U(N)^n$. Then $x = \sum x_j \otimes e_j$ is an M_N-smooth point in $M_N(OH_n)$ iff $\|T_x : H_0 \to H_0\| < n$.*

Proof. The "if part" follows from the preceding statement. Conversely, we claim that if $\|T_x : H_0 \to H_0\| = n$ then x is not an M_N-smooth point in $M_N(OH_n)$. Since this claim is unchanged if we replace x by any x' in $Orb(x)$, we may assume that $x_1 = 1$. Then if $\|T_x : H_0 \to H_0\| = n$, there is $0 \neq \xi \in H_0$ such that $\|T_x(\xi)\| = n\|\xi\|$, and hence (by the uniform convexity of Hilbert space) $x_j \xi x_j^* = x_1 \xi x_1^* = \xi$ for all j. This implies that the commutant of $\{x_j\}$ is not reduced to the scalars, and hence in a suitable basis $x_j = x_j^1 \oplus x_j^2 \in M_{N_1} \oplus M_{N_2}$ for some $N_1, N_2 \geq 1$ with $N_1 + N_2 = N$. Then the choice of $y_j = x_j^1 \oplus 0$ produces $y \in M_N^n$ not in $Orb(x)$ and such that $\|\sum x_j \otimes \bar{y}_j\| = n$. Thus x is not an M_N-smooth point in $M_N(OH_n)$, proving our claim. □

Remark 2.7. Let E be any n-dimensional operator space with a basis (e_j). Assume that for any $u = (u_j) \in U(N)^n$ we have $\|\sum u_j \otimes e_j\|_{M_N(E)} = \sqrt{n}$ and also that $\|\sum a_j \otimes e_j\|_{M_N(E)} \leq \|\sum a_j \otimes \bar{a}_j\|^{1/2}$ for all $a = (a_j) \in M_N^n$. Then, by the same proof, for any $x = (x_j) \in U(N)^n$ such that $\|T_x : H_0 \to H_0\| < n$ as above, the point $x = \sum x_j \otimes e_j$ is an M_N-smooth point in $M_N(E)$. Indeed, any y in the unit ball of $M_N(E^*)$ that M_N-norms x with respect to $M_N(E)$ is a fortiori in in the unit ball of $M_N(OH_n)$.

Lemma 1.12 above can be viewed as a refinement of this: assuming $\|T_x : H_0 \to H_0\| < \varepsilon n$ we have a certain form of "uniform" smoothness at x, the points that almost M_N-norm x up to δn are in the orbit of x up to $f(\delta)n$.

Notation: Let E be a finite dimensional operator space. Fix $C > 0$. We denote by $k_E(N, C)$ the smallest integer k such that there is a subspace F of $M_N \oplus \cdots \oplus M_N$ (with M_N repeated k-times) such that $d_N(E, F) \leq C$.

Note that for any $E \subset M_n$ we have $k_E(N, 1) = 1$ for any $N \geq n$.

The next statement is our main result in this §. It gives a lower bound for $k_E(N, C_1)$ when $E = OH_n$. We will show later (see Lemma 2.11) that a similar upper bound holds for *all* n-dimensional operator spaces. Thus for $E = OH_n$ (and also for $E = \ell_1^n$ or $E = R_n + C_n$, see Remark 2.10) the growth of $N \mapsto k_E(N, C_1)$ is essentially *extremal*.

Theorem 2.8. *There are numbers $C_1 > 1$, $b > 0$, $n_0 > 1$ and a function $n \mapsto N_0(n)$ from $\mathbb{N}$ to itself such that for any $n \geq n_0$ and $N \geq N_0(n)$, we have*

$$k_{OH_n}(N, C_1) \geq \exp bnN^2.$$

We start by recalling the classical argument dealing with the Banach space case, i.e. the case $N = 1$. Let E be an n-dimensional Banach space. Assume that, for some $C > 1$, E embeds C-isomorphically into ℓ_∞^k. For convenience we write $C = (1-\delta)^{-1}$ for some $\delta > 0$. Our embedding assumption means that there is a set $\mathcal{T}$ in the unit ball of E^* such that for any $x \in E$ we have

$$(1-\delta)\|x\| \leq \sup_{t\in\mathcal{T}} |t(x)| \leq \|x\|. \tag{2.1}$$

Then for any x in the unit ball of E, there is $t_x \in \mathcal{T}$ and $\omega_x \in \mathbb{C}$ with $|\omega_x| = 1$ such that $1 - \delta \leq \Re(\omega_x t_x(x))$.
Now assume $E = \ell_2^n$. Then identifying E and E^* as usual, we see that $1 - \delta \leq \Re(\omega_x t_x(x))$ implies $\|x - \omega_x t_x\|^2 \leq 2\delta$. In the case of real Banach spaces, $\omega_x = \pm 1$ and we conclude quickly, but let us continue for the sake of analogy with the case $N > 1$. We just proved that the set $\{\omega t \mid \omega \in \mathbb{T}, t \in \mathcal{T}\}$ is a $\sqrt{2\delta}$-net in the unit ball of $E = \ell_2^n$. Fix $\varepsilon > 0$. Let $N(\varepsilon) \approx 2\pi/\varepsilon$ be such that there is an ε-net in $\mathbb{T}$. It follows that there is a $(2\delta+\varepsilon)$-net $\mathcal{N}$ in the unit ball of $E = \ell_2^n$ with $|\mathcal{N}| \leq N(\varepsilon)|\mathcal{T}|$. But by a well known volume estimate (see e.g. [28, p. 49-50]), any δ'-net in the unit ball of $E = \ell_2^n$ must have cardinality at least $(1/\delta')^n$. Thus we conclude $(2\delta+\varepsilon)^{-n} \leq N(\varepsilon)|\mathcal{T}|$. Taking say $\varepsilon = \delta$ this yields

$$(2\pi)^{-1}3^{-n}(1/\delta)^{n-1} \leq |\mathcal{T}|.$$

Thus we find that for any $\delta < 1/3$ (actually a simple modification yields the case $\delta < 1/2$) there is a number $b > 0$ for which we obtain $|\mathcal{T}| \geq \exp bn$, and hence $k_{OH_n}(1, (1-\delta)^{-1}) \geq \exp bn$.

Remark 2.9. The preceding argument still works when E is uniformly convex with modulus $\varepsilon \mapsto \delta(\varepsilon)$. This means that if x_1, x_2 in the unit ball B_E satisfy $\|x_1 - x_2\| \geq \varepsilon$ then $\|(x_1+x_2)/2\| \leq 1 - \delta(\varepsilon)$. Indeed, the only property we used is that for any $\varepsilon > 0$ there is $r > 0$ such that $x_1, x_2 \in B_E$ and $\xi_1, \xi_2 \in B_{E^*}$ satisfy

$$\Re(\xi_1(x_1)) > 1 - r \quad \Re(\xi_2(x_2)) > 1 - r \quad \text{and} \quad \|\xi_1 - \xi_2\| < r,$$

then we must have $\|x_1 - x_2\| < \varepsilon$. To check this note that

$$\|(x_1+x_2)/2\| \geq |\xi_1(x_1+x_2)/2| \geq |\xi_1(x_1)/2 + \xi_2(x_2)/2| - \|\xi_1 - \xi_2\|/2 > 1 - r - r/2$$

thus if $r = \delta(\varepsilon)/2$ then we have $\|(x_1+x_2)/2\| > 1 - \delta(\varepsilon)$ and hence we must have $\|x_1 - x_2\| < \varepsilon$.

A completely different proof, with no restriction on δ or equivalently on the constant C can be given by a well known argument using real or complex Gaussian random variables. We restrict to the real case for simplicity. Let γ_n be the canonical Gaussian measure on $\mathbb{R}^n$. Assume (2.1). Let $q = \int \exp(x^2/4)\gamma_1(dx) < \infty$. Note that since $\mathcal{T}$ is included in the unit ball we have

$$\int \exp(\sup_{t\in\mathcal{T}} t(x)^2)\gamma_n(dx) \leq \sum_{t\in\mathcal{T}} \int \exp t(x)^2 \gamma_n(dx) \leq q|\mathcal{T}|.$$

But by (2.1), if we reset $C = 1 - \delta$, we find

$$(\int \exp(C^{-2}|x|^2)\gamma_1(dx))^n \leq \int \exp(C^2 \sum |x_j|^2)\gamma_n(dx) \leq \int \exp(\sup_{t\in\mathcal{T}} t(x)^2)\gamma_n(dx) \leq q|\mathcal{T}|.$$

Thus if we define $b = b_C > 0$ by $\int \exp(C^{-2}|x|^2)\gamma_1(dx) = \exp b$, we find $|\mathcal{T}| \geq q^{-1}\exp nb$ and we conclude

$$k_{OH_n}(1, C) \geq q^{-1}\exp b_C n.$$

Proof of Theorem 2.8. The proof follows the strategy of the first proof outlined above for $N = 1$, but using Theorem 1.5 instead of the lower bound on the metric entropy of the unit ball of ℓ_2^n. Consider an n-dimensional operator space E. Let $k = k_E(N, C)$. Let again $C = (1-\delta)^{-1}$. Then there is a set $\mathcal{T}$ with $|\mathcal{T}| = k$ and completely contractive mappings $\phi_t : E \to M_N$ such that

$$\forall x \in M_N(E) \quad (1-\delta)\|x\|_{M_N(E)} \le \sup_{t\in\mathcal{T}} \|(\phi_t)_N(x)\|_{M_N(M_N)}. \tag{2.2}$$

Let e_j be a basis for E so that each x can be developed as $x = \sum x_j \otimes e_j \in M_N \otimes E$. Let $y(t) \in M_N(E^*)$ be the element associated to $\phi_t : E \to M_N$. Let $e_j^+ \in E^*$ be the basis of E^* that is biorthogonal to (e_j). Then $y(t)$ (or equivalently ϕ_t) can be written as $y(t) = \sum y_j(t) \otimes e_j^+ \in M_N \otimes E^*$. Then (2.2) can be rewritten as:

$$\forall x \in M_N(E) \quad (1-\delta)\|x\|_{M_N(E)} \le \sup_{t\in\mathcal{T}} \|\sum x_j \otimes y_j(t)\|_{M_N(M_N)}.$$

Moreover each $y(t)$ is in the unit ball of $M_N(E^*) = CB(E, M_N)$. We now assume $E = OH_n$. Fix $\varepsilon > 0$ (to be determined later). Let us denote by T_ε the set appearing in Theorem 1.5. This gives us

$$\forall x = (x_j) \in T_\varepsilon \quad (1-\delta)n^{1/2} \le \sup_{t\in\mathcal{T}} \|\sum x_j \otimes y_j(t)\|_{M_N(M_N)}.$$

Let $\bar{v}(t) = (\bar{v}_j(t)) \in M_N^n$ be associated to $\sqrt{n}y(t)$, so that we have

$$\forall x = (x_j) \in T \quad (1-\delta)n \le \sup_{t\in\mathcal{T}} \|\sum x_j \otimes \bar{v}_j(t)\|_{M_N(M_N)}.$$

For any $x \in T_\varepsilon$ there is a point $t_x \in \mathcal{T}$ such that

$$(1-\delta)n \le \|\sum x_j \otimes \bar{v}_j(t_x)\|.$$

Let $v_x = (v_j(t_x))$. By Lemma 1.12, the last inequality implies $d'(x, v_x) < f(\delta')\sqrt{n}$ for any $\delta' > \delta$, and hence $d'(x, v_x) \le f(\delta)\sqrt{n}$. Moreover by the second (much easier) part of Lemma 1.12, we know that $d'(x, y) \ge \sqrt{2\delta_0 n}$ for any $x \ne y \in T_\varepsilon$, since x, y are δ_0-separated. We claim that after suitably adjusting the parameters we have $|T_\varepsilon| \le |\mathcal{T}|$. Indeed, assume that $|T_\varepsilon| > |\mathcal{T}|$, then there must exist $x \ne y \in T_\varepsilon$ such that $v_x = v_y$. We have then

$$\sqrt{2\delta_0 n} \le d'(x, y) \le d'(x, v_x) + d'(v_x, y) = d'(x, v_x) + d'(v_y, y) \le 2f(\delta)\sqrt{n}$$

and hence $\sqrt{2\delta_0} \le 2f(\delta)$. But we can clearly choose $\varepsilon > 0$ and $\delta = \delta_1 > 0$ small enough so that $2f(\delta_1) < \sqrt{2\delta_0}$, so this contradiction proves our claim that $|T_\varepsilon| \le |\mathcal{T}|$, and hence $|\mathcal{T}| \ge \exp \beta n N^2$ at least for $\delta = \delta_1 > 0$. Let $C_1 = (1-\delta_1)^{-1}$. Thus we have proved $k_{OH_n}(N, C_1) \ge \exp \beta n N^2$. □

Remark 2.10. Let E be any n-dimensional operator space with a basis (e_j). Assume that there is a scaling factor $c > 0$ (that does not play any role in the estimate) such that for any $u = (u_j) \in U(N)^n$ we have $c\|\sum u_j \otimes e_j\|_{M_N(E)} = \sqrt{n}$ and also that $c\|\sum a_j \otimes e_j\|_{M_N(E)} \le \|\sum a_j \otimes \bar{a}_j\|^{1/2}$ for all $a = (a_j) \in M_N^n$. Then, arguing as in Remark 2.7, we find $k_E(N, C_1) \ge \exp \beta n N^2$. This shows that this estimate is valid for $R_n + C_n$ (take $c = 1$) and for ℓ_1^n equipped with its maximal operator space structure (take $c = n^{-1/2}$).

We now turn to the reverse inequality to that in Theorem 2.8. This general estimate is easy to check by a rather routine argument.

Lemma 2.11. *Let E be an n-dimensional operator space, then for any $0 < \delta < 1$ we have*

$$k_E(N, (1-\delta)^{-1}) \leq (1 + 2\delta^{-1})^{2nN^2}.$$

Therefore, for any operator space X, any finite dimensional subspace $E \subset X$ we have

$$\forall C > 1 \quad \limsup_{N\to\infty} \frac{\log k_E(N,C)}{N^2} < \infty.$$

Proof. Let $x \in M_N(E)$ and let $\hat{x} : E^* \to M_N$ denote the associated linear mapping. Recall $\|x\| = \|\hat{x}\|_{cb}$. By Lemma 2.2 $\|\hat{x}\|_{cb} = \sup\{\|(\hat{x})_N(y)\|_{M_N(M_N)} \mid y \in B_N\}$ where we denote here by B_N the unit ball of $M_N(E^*)$ viewed as a real space. Since the latter ball is $2nN^2$-dimensional, it contains a δ-net $\{y_i \mid i \leq m\}$ with cardinality $m \leq (1 + 2\delta^{-1})^{2nN^2}$ (see e.g. [28, p. 49-50]). By an elementary estimate, we have then (for any $x \in M_N(E)$)

$$\sup_{i\leq m} \|(\hat{x})_N(y_i)\| \leq \|\hat{x}\|_{cb} = \|x\| \leq (1-\delta)^{-1} \sup_{i\leq m} \|(\hat{x})_N(y_i)\|. \tag{2.3}$$

Let $u : E \to \oplus_{i\leq m} M_N$ be the mapping defined by (here again $\hat{y}_i : E \to M_N$ is associated to y_i)

$$u(e) = \oplus_{i\leq m} \hat{y}_i(e)$$

for any $e \in E$. Let $F \subset \oplus_{i\leq m} M_N$ be the range of u. Then (2.3) says that $\|u_N\| \leq 1$ and $\|u_N^{-1}\| \leq 1 + \delta$, and hence $d_N(E, F) \leq (1-\delta)^{-1}$. Thus $k_E(N, (1-\delta)^{-1}) \leq m$. □

Definition 2.12. An operator space X will be called matricially C-subGaussian if

$$\limsup_{N\to\infty} \frac{\log k_E(N,C)}{N^2} = 0.$$

for any finite dimensional subspace $E \subset X$. We say that X is matricially subGaussian if it is matricially C-subGaussian for some $C \geq 1$. (See Remark 3.2 for the reason behind "matricially").
Note: If X itself is finite dimensional, it suffices to consider $E = X$.
We will denote by $C_g(X)$ the smallest C such that X is matricially C-subGaussian.

The preceding result (resp. Remark 2.10) shows that when $C < C_1$, then OH (resp. ℓ_1 or $R + C$) is not matricially C-subGaussian. In sharp contrast, any C-exact operator space (we recall the definition below) E is clearly matricially C-subGaussian since, for any $c > C$, it satisfies $k_E(N, c) = 1$ for all N large enough. We do not know whether conversely the latter property implies that E is C-exact (but we doubt it).

3. Random matrices and subexponential operator spaces

Our goal is to study a generalization of the notion of exact operator space for which the version of Grothendieck's theorem obtained in [35] is still valid.
Notation: Let E be a finite dimensional operator space. Fix $C > 0$. We denote by $K_E(N, C)$ the smallest integer K such that there is an operator subspace $F \subset M_K$ such that

$$d_N(E, F) \leq C.$$

Definition 3.1. We say that an operator space X is C-subexponential if

$$\limsup_{N\to\infty} \frac{\log K_E(N,C)}{N} = 0,$$

for any finite dimensional subspace $E \subset X$. We say that X is subexponential if it is C-subexponential for some $C \geq 1$.
Note: If X itself is finite dimensional, it suffices to consider $E = X$.
We will denote by $C(X)$ the smallest C such that X is C-subexponential.

Remark 3.2. In the same vein, it is natural to call an operator space X C-subGaussian if $\limsup_{N\to\infty} N^{-2}\log K_E(N,C) = 0$ for any finite dimensional subspace $E \subset X$. We do not have significant information about this class at this point, but to avoid confusion, we decided to call "matricially subGaussian" the spaces in Definition 2.12. Clearly "matricially subGaussian" implies "subGaussian" but the converse is unclear.

Recall that an operator space X is called C-exact if for any finite dimensional subspace $E \subset X$ and any $c > C$ there is a k and $F \subset M_k$ such that $d_{cb}(E,F) < c$. We denote by $ex(X)$ the smallest such C. We say that X is exact if it is C-exact for some $C \geq 1$. As shown by Kirchberg, a C^*-algebra is exact iff it is 1-exact. See [30, ch.17] or [2] for more background on exactness.

Lemma 3.3. *An operator space X is C-exact iff*

$$\forall c > C \quad \sup_{N\geq 1} K_E(N,c) < \infty.$$

for any finite dimensional subspace $E \subset X$.

Proof. The only if part is obvious since $d_N \leq d_{cb}$. Conversely, assume that for some fixed k we have

$$\sup_{N\geq 1} K_E(N,c) \leq k.$$

We have then for each N a subspace $F_N \subset M_k$ and a mapping $u(N) : \; E \to F_N$ such that $\|u(N)_N\| \leq c + 1/N$ and $\|u(N)^{-1}{}_N\| \leq 1$. Let F be an ultraproduct of (F_N) along a free ultrafilter (see e.g. [30] for ultraproducts of operator spaces), and let $u : \; E \to F$ be the mapping associated to $(u(N))$. Then clearly $\|u\|_{cb} \leq c$ and $\|u^{-1}\|_{cb} \leq 1$. So we obtain $d_{cb}(E,F) \leq c$ and F obviously embeds completely isometrically into M_k. □

We will denote by $Y^{(N)}$ a random $N \times N$-matrix with i.i.d. complex Gaussian entries with mean zero and second moment equal to $N^{-1/2}$, and we denote by $(Y_j^{(N)})$ a sequence of i.i.d. copies of $Y^{(N)}$. The following result follows from the main estimates in §2 in [11].

Theorem 3.4 ([11])**.** *For any $a_1, \cdots, a_n \in B(H)$, let $S = \sum_1^n a_j \otimes Y_j^{(N)}$.*
Assume that $\sum a_j^* a_j \leq 1$ and $\sum a_j a_j^* \leq 1$, or equivalently $\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\} \leq 1$.
Let $\Sigma \in B(H)$ be defined by the identity $\mathbb{E}(S^*S)^p = \Sigma \otimes 1$. For any positive integer p, we have

$$\mathbb{E}(S^*S)^p = \Sigma \otimes 1 \leq 1 \otimes \mathbb{E}(Y^{(N)^*} Y^{(N)})^p.$$

Consequently, if $a_1, \cdots, a_n$ are such that $\mathrm{tr}|a_j|^{2p} < \infty$ and $\dim(H) = k$ we have

$$(\mathbb{E}\ \mathrm{tr}|S|^{2p})^{1/2p} \leq (k\mathbb{E}\ \mathrm{tr}|Y^{(N)}|^{2p})^{1/2p}. \tag{3.1}$$

where the trace on the left is on $H \otimes \ell_2^N$ and the one on the right is on ℓ_2^N.

This combines Prop. 2.5 and Prop. 2.7 in [11]. Note that the formula given for Σ in Prop. 2.5 in [11] is such that if $a_1 = \cdots = a_n = n^{-1/2}$ the corresponding Σ must be equal to $\mathbb{E}(Y^{(N)^*}Y^{(N)})^p$. So the preceding inequality is implicitly in [11].

Remark 3.5. Actually, using [3] combined with Prop. 2.5 in [11], it is easy to see more generally that if $\max\{\|(\sum a_j^* a_j)^{1/2}\|_{2p}, \|(\sum a_j a_j^*)^{1/2}\|_{2p}\} \le 1$, then

$$\mathbb{E}\|S\|_{2p}^{2p} \le \mathbb{E}\|Y^{(N)}\|_{2p}^{2p}.$$

Note that this bound is obviously best possible. It can be interpreted as a sort of “Khintchine inequality” for Gaussian random matrices with best possible constant.

Remark 3.6. In sharp contrast, we do not know whether the constant $C(N, p)$ in the following variant of this Khintchine inequality is significantly better (as a function of N) than $\mathbb{E}\|Y^{(N)}\|_{2p} \approx (1 + (p/N)^{1/2})$: We denote by $C(N, p)$ the smallest constant C such that for any finite sequence a_j in S_p there are $N \times N$ unitary matrices y_j such that

$$\|\sum a_j \otimes y_j\|_{L_{2p}(tr\times\tau_N)} \le C \max\{\|(\sum a_j^* a_j)^{1/2}\|_{2p}, \|(\sum a_j a_j^*)^{1/2}\|_{2p}\}. \tag{3.2}$$

Note that there is a universal constant c such that if $(U_j^{(N)})$ are i.i.d. random variables uniformly distributed over the unitary group $U(N)$ we have (for any a_j in S_{2p})

$$\|\sum_1^n a_j \otimes U_j^{(N)}\|_{2p} \le c\|\sum_1^n a_j \otimes Y_j^{(N)}\|_{2p}.$$

Indeed this can be shown by writing the polar decomposition of $Y_j^{(N)}$ (say as $Y_j^{(N)} = U_j^{(N)}|Y_j^{(N)}|$) and taking the conditional expectation with respect to the unitary parts of $Y_j^{(N)}$. From this follows that $C(N, p) \le c'(1 + (p/N)^{1/2})$. A significant improvement, such as for instance $C(N, p) \le c(1 + p^{1/2}/N)$, would narrow (or close) the gap appearing when one compares with Lemma 2.11. Indeed, note that we could use (3.2) in place of (3.3) in Lemma 5.2, and an estimate such as $C(N, p) \le c(1 + p^{1/2}/N)$ would then lead to $K_E(N, C) \ge \exp \delta N^2 d$ for $E = \ell_1^d$ with maximal operator space structure. Note that for $N = 1$ the order of growth of $C(N, p)$ in $p^{1/2}$ clearly is optimal (e.g. because the y_j's can be absorbed when the a_j's are sign invariant and we can invoke the central limit theorem), but, at the time of this writing, we do not see why its dependence in N cannot be improved.

We will use the following direct consequence of Theorem 3.4 and concentration of measure.

Lemma 3.7. *For any $\varepsilon > 0$, there is a constant γ_ε and a number $N(\varepsilon)$ such that for any $N \ge N(\varepsilon)$, any integer k and any $a_1, \cdots, a_n \in M_k$ we have*

$$\mathbb{E}\left\|\sum\nolimits_1^n a_j \otimes Y_j^{(N)}\right\| \le (1+\varepsilon)\left(2 + \gamma_\varepsilon(\frac{\log(k) + 1}{N})^{1/2}\right) \max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}. \tag{3.3}$$

Proof. Let X be any Gaussian random variable with values in a (real) Banach space B. Let

$$\sigma(X) = \sup\{(\mathbb{E}|\xi(X)|^2)^{1/2} \mid \xi \in B^*, \|\xi\| \le 1\}$$

It will be convenient to use the following concentration of measure inequality (see [32] for a very simple proof):

$$\|\|X\| - \mathbb{E}\|X\|\|_p \le (\pi/2)\sigma(X)\|g\|_p,$$

where g is a standard Gaussian normal random variable, and this implies

$$(\mathbb{E}\|X\|^p)^{1/p} \le \mathbb{E}\|X\| + (\pi/2)\sigma(X)\|g\|_p. \tag{3.4}$$

We will view $Y^{(N)}$ as B-valued with $B = M_N$ considered as a real Banach space. We have then

$$\sigma(Y^{(N)}) \le N^{-1/2}.$$

Thus the preceding inequality applied to $X = Y^{(N)}$ yields by (3.4)

$$(\mathbb{E}\|Y^{(N)}\|_{2p}^{2p})^{1/2p} \le (N)^{1/2p}(\mathbb{E}\|Y^{(N)}\|^{2p})^{1/2p} \le (N)^{1/2p}(\mathbb{E}\|Y^{(N)}\| + (\pi/2)\sigma(Y^{(N)})\|g\|_{2p}).$$

It is well known that $\lim_{N\to\infty} \mathbb{E}\|Y^{(N)}\|_{M_N} = 2$. In fact we need only an upper bound, so we set

$$\varepsilon(N) = \mathbb{E}\|Y^{(N)}\| - 2 \quad \text{and we note} \quad \lim_{N\to\infty} \varepsilon(N) = 0.$$

Since there is a constant β such that $\|g\|_{2p} \le \beta\sqrt{2p}$ for all $p \ge 1$, we find

$$(\mathbb{E}\|Y^{(N)}\|_{2p}^{2p})^{1/2p} \le (N)^{1/2p}(2 + \varepsilon(N) + \beta(\pi/2)(2p/N)^{1/2}).$$

We also have obviously $\|S\| \le \|S\|_{2p}$ for any $p \ge 1$ and hence

$$(\mathbb{E}\|S\|^{2p})^{1/2p} \le (\mathbb{E}\|S\|_{2p}^{2p})^{1/2p}.$$

By homogeneity we may assume $\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\} \le 1$. By (3.1) this gives us

$$(\mathbb{E}\|S\|^{2p})^{1/2p} \le (kN)^{1/2p}(2 + \varepsilon(N) + \beta(\pi/2)(2p/N)^{1/2}). \tag{3.5}$$

Fix $0 < \varepsilon \le 1$. For $N \ge N(\varepsilon)$ we have

$$(\mathbb{E}\|S\|^{2p})^{1/2p} \le (kN)^{1/2p}(2 + \varepsilon/2 + \beta(\pi/2)(2p/N)^{1/2}). \tag{3.6}$$

Choose p large enough so that $(kN)^{1/2p} = 1 + \varepsilon/2$, so that $p \approx \varepsilon^{-1}(\log(kN))$. For some numerical constant β', we obtain

$$\mathbb{E}\|S\| \le (\mathbb{E}\|S\|^{2p})^{1/2p} \le (1 + \varepsilon/2)(2 + \varepsilon/2 + \beta'(\varepsilon^{-1}\log(kN)/N)^{1/2}), \tag{3.7}$$

and this leads to (3.3). $\square$

This leads us to

Theorem 3.8. *Let E be a C-subexponential operator space. Then for any $a_1, \cdots, a_n \in E$ we have*

$$\limsup_{N\to\infty} \mathbb{E}\left\|\sum_1^n a_j \otimes Y_j^{(N)}\right\| \le 2C \max\left\{\left\|(\sum a_j^* a_j)^{1/2}\right\|, \left\|(\sum a_j a_j^*)^{1/2}\right\|\right\}. \tag{3.8}$$

Proof. Replacing E by the span of the a_j's, we may clearly assume E finite dimensional. Fix $c > C$. Consider $u:\ E \to F$ with $F \subset M_k$, $k = K_E(N, C)$ and $\|u_N\|\|u^{-1}{}_N\| \le c$.
By homogeneity we may assume $\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\} \le 1$. Let $b_j = u(a_j)$. We may assume $n \le N$. Then we have

$$\max\{\|(\sum b_j^* b_j)^{1/2}\|, \|(\sum b_j b_j^*)^{1/2}\|\} \le \|u_n\| \le \|u_N\|,$$

and also

$$\|\sum\nolimits_1^n a_j \otimes Y_j^{(N)}\| \le \|u^{-1}{}_N\| \|\sum_1^n b_j \otimes Y_j^{(N)}\|.$$

By (3.3) (applied with b_j in place of a_j) this gives us

$$\mathbb{E}\|\sum\nolimits_1^n a_j \otimes Y_j^{(N)}\| \le \|u^{-1}{}_N\| \|u_N\|(1+\varepsilon)(2+\gamma_\varepsilon\left(\frac{\log(k)+1}{N}\right)^{1/2}),$$

and therefore

$$\limsup\nolimits_{N\to\infty} \mathbb{E}\|\sum\nolimits_1^n a_j \otimes Y_j^{(N)}\| \le 2c(1+\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, this concludes the proof. □

Remark 3.9. By in [11, Th. 3.3], with the same notation as in the above Theorem 3.4, assuming $\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\} \le 1$ we have for any $0 \le t \le N/2$

$$\mathbb{E} \exp t\|S\|^2 \le kN \exp(4t + 4t^2/N).$$

By convexity this implies $\exp t(\mathbb{E}\|S\|)^2 \le kN \exp(4t + 4t^2/N)$, and hence taking the log we find

$$\mathbb{E}\|S\| \le 2\left(1 + t/N + (4t)^{-1}\log(kN)\right)^{1/2}$$

from which taking $t = [\varepsilon N]$ for $\varepsilon < 1/2$ it is easy to deduce (3.3). Note however that the deduction of [11, Th. 3.3] from Prop. 2.5 and Prop. 2.7 in [11] involves rather heavy calculations, and that explains why we presented the above short cut (based only on Prop. 2.5 and Prop. 2.7 in [11]) using concentration of measure instead of invoking [11, Th. 3.3]. In addition, this route allowed us to draw the reader's attention to Buchholz's nice contribution [3] in Remark 3.5.

Remark 3.10. Note for future reference that with the preceding notation for any $a_1, \cdots, a_n \in E$, any $\varepsilon > 0$ and any $N \ge N(\varepsilon), C$ we have

$$\mathbb{E}\|\sum\nolimits_1^n a_j \otimes Y_j^{(N)}\| \le C(1+\varepsilon)\left(2+\gamma_\varepsilon(\frac{\log(K_E(N,C)+1}{N})^{1/2}\right)\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}.$$

Remark 3.11. Let $\mathcal{N}$ be a finite subset with cardinality $\le m$ of the set of all n-tuples $a = (a_j)$ in M_k such that $\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\} \le 1$. Let $S_a = \sum_1^n a_j \otimes Y_j^{(N)}$. We claim that there is a numerical constant β'' such that for any $0 < \varepsilon \le 1$ and $N \ge N(\varepsilon)$ we have

$$\mathbb{E} \sup_{a\in\mathcal{N}} \|S_a\| \le (1+\varepsilon)(2+\beta''(\varepsilon^{-1}\log(mkN)/N)^{1/2}), \tag{3.9}$$

and moreover that

$$\mathbb{P}\{\sup_{a\in\mathcal{N}} \|S_a\| \ge (1+\varepsilon)^2(2+\beta''(\varepsilon^{-1}\log(mkN)/N)^{1/2})\} \le (mkN)^{-1}. \tag{3.10}$$

Indeed, we have obviously

$$\mathbb{E}\sup_{a\in\mathcal{N}}\|S_a\| \le (\mathbb{E}\sup_{a\in\mathcal{N}}\|S_a\|^{2p})^{1/2p} \le (\mathbb{E}\sum\nolimits_{a\in\mathcal{N}}\|S_a\|^{2p})^{1/2p} \le m^{1/2p}\sup_{a\in\mathcal{N}}(\mathbb{E}\|S_a\|^{2p})^{1/2p},$$

and hence by (3.6)

$$(\mathbb{E}\sup_{a\in\mathcal{N}}\|S_a\|^{2p})^{1/2p} \le (mkN)^{1/2p}(2+\varepsilon/2+\beta(\pi/2)(2p/N)^{1/2}).$$

Let $\lambda = (mkN)^{1/2p}(2+\varepsilon/2+\beta(\pi/2)(2p/N)^{1/2})$. A fortiori by Tshebyshev's inequality

$$\mathbb{P}\{\sup_{a\in\mathcal{N}}\|S_a\| \geq \lambda(mkN)^{1/2p}\} \leq (mkN)^{-1}.$$

We now repeat the preceding step: Choose p large enough so that $(mkN)^{1/2p} = 1+\varepsilon/2$, so that $p \approx \varepsilon^{-1}(\log(mkN))$. For some numerical constant β'', we obtain the announced claims. □

The preceding Remark will be used to construct the examples in the last section via the following consequence:

Theorem 3.12. *For any $0 < \varepsilon < 1$ there is a constant $c_\varepsilon > 0$ such that whenever $N \geq c_\varepsilon nk^2$ then with probability greater than $1-(3^{2nk^2}kN)^{-1}$ we have*

$$\forall (a_j) \in M_k^n \quad \|S_a\| \leq (2+\varepsilon)\max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}.$$

Proof. Let Z be any normed space of real dimension d. It is well known (see e.g. [28, p. 49-50]) that there is an ε-net $\mathcal{N}_\varepsilon$ in the unit ball of the space Z with cardinal $\leq (1+2/\varepsilon)^d$. Let Z be M_k^n equipped with the norm $\|a\| = \max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}$. Note $d = 2nk^2$. It is well known that for any linear map such as $a \mapsto S_a$ we have

$$\sup_{a\in B_Z}\|S_a\| \leq (1-\varepsilon)^{-1}\sup_{a\in\mathcal{N}_\varepsilon}\|S_a\|.$$

Thus the result follows from (3.10) with $m = (1+2/\varepsilon)^{2nk^2}$. □

Notation: For $a = (a_1, \cdots, a_n)$ with $a_j \in E$ we denote

$$\|a\|_{RC} = \max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\},$$

$$\|a\|_R = \|(\sum a_j a_j^*)^{1/2}\|, \text{ and } \|a\|_C = \|(\sum a_j^* a_j)^{1/2}\|,$$

so that

$$\|a\|_{RC} = \max\{\|a\|_R, \|a\|_C\}.$$

By the same proof as in [16] (see also [30, Th. 19.1]), we can now state

Corollary 3.13. *Let E, F be subexponential operator spaces with respective constants $C(E), C(F)$. Then any c.b. linear map $u:\ E \to F^*$ satisfies for any n, any $a = (a_1, \cdots, a_n) \in E^n$ and any $b = (b_1, \cdots, b_n) \in F^n$*

$$|\sum \langle u(a_j), b_j\rangle| \leq 4C(E)\ C(F)\|u\|_{cb}\|a\|_{RC}\|b\|_{RC}.$$

It is easy to check that if X is C-subexponential, the minimal tensor product $K(\ell_2)\otimes_{\min} X$ (of X with the set $K(\ell_2)$ of all compact operators on ℓ_2) is also C-subexponential. Indeed, by a perturbation argument we may restrict to finite dimensional subspaces of the form $M_n(E)$ with $E \subset X$. Then we have obviously

$$K_{M_n(E)}(N, C) \leq nK_E(nN, C), \tag{3.11}$$

and hence the subexponential character is preserved, Therefore by the same method as in the second proof given in [33, §18] of GT for exact operator spaces, using the Haagerup-Musat ideas from [13] or the very recent much simpler proof by Regev and Vidick [36], and combining that with [16], we obtain:

Corollary 3.14. *Let E, F be subexponential operator spaces with respective constants $C(E), C(F)$. Then any c.b. linear map $u: E \to F^*$ satisfies for any n, any $a = (a_1, \cdots, a_n) \in E^n$, any $b = (b_1, \cdots, b_n) \in F^n$ and any $t_j > 0$*

$$|\sum \langle u(a_j), b_j\rangle| \le 4C(E)\ C(F)\|u\|_{cb}(\|(a_j)\|_R\|(b_j)\|_C + \|(t_ja_j)\|_C\|(t_j^{-1}b_j)\|_R).$$

Remark 3.15. The preceding proofs suggest that perhaps one should keep track of the dependence in E in studying spaces like subexponential ones. One possibility would be to define X as (C, C')-subexponential if for any finite dimensional $E \subset X$ we have

$$\limsup_{N\to\infty} N^{-1}\log K_E(N, C) \le C'.$$

Note however that the constant C' does not seem to behave as well as C (see (3.11)) when one passes from E to $M_n(E)$.

Remark 3.16. Given an operator space X, it is natural to introduce the following parameters:

$$k_X(N, C; d) = \sup\{k_E(N, C) \mid E \subset X,\ \dim(E) = d\}$$

$$K_X(N, C; d) = \sup\{K_E(N, C) \mid E \subset X,\ \dim(E) = d\}.$$

We will say that X is uniformly subexponential (resp. uniformly matricially subGaussian) if there is C such that

$$\forall d \ge 1 \quad \limsup_{N\to\infty} \frac{\log K_X(N, C; d)}{N} = 0 \quad (\text{resp. } \limsup_{N\to\infty} \frac{\log k_X(N, C; d)}{N^2} = 0).$$

Similarly we will say that X is uniformly exact if there is C such that for all d

$$\sup\{K_X(N, C; d) \mid N \ge 1\} < \infty.$$

It is easy to check that if X is uniformly exact (resp. uniformly subexponential, rresp. uniformly matricially subGaussian) then all ultrapowers of X are exact (resp. subexponential, rresp. matricially subGaussian). Note however (I am indebted to Yanqi Qiu for this remark) that the converse is unclear.
For example, R or C (or $R \oplus C$), any commutative C^* algebra A, or any space of the form $A \otimes_{\min} M_N$ is uniformly exact. It would be interesting to characterize uniformly exact operator spaces.

Remark 3.17. It is tempting to weaken the definition of subexponential spaces by replacing the limsup there by a liminf. Such spaces could be called weakly subexponential. We do not know whether this is a true weakening. Then Corollaries 3.13 and 3.14 extend to the case when one of E, F is weakly subexponential and the other one subexponential. Note however that, a priori, the case when both E, F are weakly subexponential is unclear.

Problems:
1) Let $C > 1$. Assume that a finite dimensional space E satisfies $k_E(N, C) \le 1$ for all N. What does that imply on E ? Is E exact ?
2) Assume E subexponential. What growth does that imply for $k_E(N, C)$? Is E uniformly matricially subGaussian ?
3) What is the order of growth (when $N \to \infty$) of $\log K_E(N, C)$ for $E = \ell_1^n$ or $E = OH_n$? In particular, when C is close to 1, is it $O(N)$?

4. Large constants of subexponentiality

We will now examine some examples. It turns out that the most commonly known non-exact operator spaces are also not subexponential, and the associated constants have a similar growth.

We start by discussing maximal operator spaces. (See e.g. [30] for the definitions of minimal and maximal operator spaces.)

Proposition 4.1. *Let E be any n-dimensional space with its maximal operator space structure. Then*

$$C(E) \geq c\sqrt{n}$$

where $c > 0$ is a constant independent of n.

Proof. We transplant from exact to subexponential an argument from [16]. Note that $C(E^*) = 1$ since E^* is a minimal operator space. By Corollary 3.13 (with $F^* = E$ and u the identity of E) we have for all finite sequences (a_j, b_j) in $E \times E^*$

$$|\sum \langle a_j, b_j \rangle| \leq 4C(E)\|a\|_{RC}\|b\|_{RC}$$

but here $\|b\|_{RC} = \sup\{(\sum |b_j(x)|^2)^{1/2} \mid x \in E, \|x\| \leq 1\}$ and $\|a\|_{RC} \leq (\sum \|a_j\|^2)^{1/2}$, so this implies

$$|\sum \langle a_j, b_j \rangle| \leq 4C(E)(\sum \|a_j\|^2)^{1/2} \sup\{(\sum |b_j(x)|^2)^{1/2} \mid x \in E, \|x\| \leq 1\}$$

and hence

$$(\sum \|b_j\|^2)^{1/2} \leq 4C(E) \sup\{(\sum |b_j(x)|^2)^{1/2} \mid x \in E, \|x\| \leq 1\}.$$

Equivalently, this means the 2-summing norm $\pi_2(E)$ of the identity of E is $\leq 4C(E)$. But it is well known (see e.g. [28, p. 35]) that $\pi_2(E) = \sqrt{n}$. Thus we conclude $C(E) \geq \sqrt{n}/4$. □

Remark 4.2. In the converse direction, for any n-dimensional operator space E we have $C(E) \leq ex(E)$ and it is known (see [30, Cor. 7.7 p. 133]) that $ex(E) \leq \sqrt{n}$.

Remark 4.3. We claim that

$$n^{1/4}/2 \leq C(OH_n) \leq n^{1/4}.$$

Indeed, applying (3.8) with $E = OH_n$ (see [30, §7]) and with a_j an orthonormal basis we find

$$\limsup_{N\to\infty} \mathbb{E}\left(\left\|\sum_1^n Y_j^{(N)} \otimes \overline{Y_j^{(N)}}\right\|^{1/2}\right) \leq 2C(OH_n)n^{1/4},$$

and since $\left\|\sum_1^n Y_j^{(N)} \otimes \overline{Y_j^{(N)}}\right\| \geq \sum_1^n \tau_N |Y_j^{(N)}|^2$ and (by the law of large numbers) $\sum_1^n \tau_N |Y_j^{(N)}|^2 \approx n$ we obtain

$$n^{1/2} = \limsup_{N\to\infty} \mathbb{E}\left((\sum_1^n \tau_N |Y_j^{(N)}|^2)^{1/2}\right) \leq 2C(OH_n)n^{1/4}$$

and hence $C(OH_n) \geq n^{1/4}/2$. In the converse direction, we have $C(OH_n) \leq ex(OH_n)$ and it is known (see [30, (10.8) p. 219]) that $ex(OH_n) \leq n^{1/4}$.

Remark 4.4. We claim that

$$n^{1/2}/2 \leq C(R_n + C_n) \leq n^{1/2}.$$

Indeed, applying (3.8) with $E = R_n + C_n$ (see [30, §2.7]) and with a_j an orthonormal basis we find similarly (since $\|(a_j)\|_{RC} = 1$)

$$n^{1/2} = \limsup_{N\to\infty} \mathbb{E}\left((\sum_1^n \tau_N |Y_j^{(N)}|^2)^{1/2} \right) \le 2C(R_n + C_n),$$

and thus we obtain

$$C(R_n + C_n) n^{-1/2} \ge 1/2.$$

In the converse direction, by Remark 4.2 we have $C(R_n + C_n) \le n^{1/2}$.

5. More growth estimates

Since $K_E(N, 1+\delta) \le N k_E(N, 1+\delta)$, the following is an immediate consequence of Lemma 2.11:

Lemma 5.1. *Let E be a n-dimensional operator space, then for any $\delta > 0$ we have*

$$K_E(N, 1+\delta) \le N(1 + 2\delta^{-1})^{2nN^2}.$$

Therefore, for any operator space X, any finite dimensional subspace $E \subset X$ we have

$$\forall C > 1 \quad \limsup_{N\to\infty} \frac{\log K_E(N, C)}{N^2} < \infty.$$

In [25] the parameter denoted below by $n(E, c)$ was introduced for an n-dimensional Banach space E and a constant c (in [25] we fixed $c = 2$). We denote by $n(E, c)$ the smallest k such that E can be embedded c-isomorphically into ℓ_∞^k. In Banach space theory Gaussian random variables can be used to give a quick proof of the fact that if either $E = \ell_2^n$ or $E = \ell_1^n$ then there is $\delta = \delta_c > 0$ such that $n(E, c) \ge \exp(\delta n)$. In [25] an estimate due to Maurey is presented showing that this remains true (with $\delta = \delta(c, c') > 0$) whenever E^* has type $p > 1$ with constant at most c'.
The problem of estimating $k_E(N, C)$ is entirely analogous to the one considered in [25] for $n(E, c)$. More precisely, we have simply $n(E, C) = k_E(1, C)$.

The preceding inequality (3.3) allows us, in the next Lemma, to prove analogous results, for some operator spaces. Note however that in the Banach space case (equivalently in the case $N = 1$), we also know that $n(E, c) \le \exp(\delta' n)$ for some universal constant δ', but in the operator space case, all we have so far in the same direction is Lemma 5.1 (see however the speculations in Remark 3.6).

Lemma 5.2. *If E is ℓ_1^n equipped with its maximal operator space structure, then for any $C > 1$ there are an integer d_0 and $\delta > 0$ depending only on C such that for any $n \ge d_0, N \ge 1$ we have*

$$K_E(N, C) \ge \exp \delta N n \tag{5.1}$$

If $E = R_n + C_n$ or ℓ_2^n equipped with its maximal operator space structure (resp. $E = OH_n$), this still holds (resp. we have $K_E(N, C) \ge \exp \delta N n^{1/2}$) for all $N \ge n$.

Proof. With the notation in Theorem 3.4, let a_j be the canonical basis of ℓ_1^n (resp. OH_n, rresp. $R_n + C_n$). Then it is easy to check on the one hand that $\mathbb{E}\|S\| \ge \alpha n$ (resp. rresp. $\mathbb{E}\|S\| \ge \alpha n^{1/2}$) for some $\alpha > 0$. On the other hand $\|a\|_{RC} = \max\{\|(\sum a_j^* a_j)^{1/2}\|, \|(\sum a_j a_j^*)^{1/2}\|\}$ is equal to $n^{1/2}$

(resp. $n^{1/4}$, rresp. 1). Thus using (3.3) and the same reasoning as in the proof of Theorem 3.8 (note that we can use $\|u\|_{cb} = \|u\|$ when u is defined on a maximal space), we find if $E = \ell_1^n$

$$\alpha n \le C(1+\varepsilon)\left(2 + \gamma_\varepsilon(\frac{\log(k)+1}{N})^{1/2}\right) n^{1/2},$$

from which we deduce for n large enough

$$(\alpha/C(1+\varepsilon))n^{1/2} \approx (\alpha/C(1+\varepsilon))n^{1/2} - 2 \le \gamma_\varepsilon(\frac{\log(k)+1}{N})^{1/2},$$

which is the announced lower bound (taking e.g. $\varepsilon = 1$). The cases $E = OH_n$ and $E = R_n + C_n$ are similar. When (a_j) is the basis of $E = \ell_2^n$ with its maximal operator space structure, by a well known result (see Exercise 28.1 in [30]) we have also a lower bound $\mathbb{E}\|S\| \ge \alpha n$ provided $n \le N$. In this case, the remaining estimate of $\|a\|_{RC}$ required to complete the proof can be found in [30, p. 223]. □

6. Examples of non-exact subexponential spaces

The original idea for our construction is as follows. As described in [16] or in [30, ch. 21], results such as Theorem 3.12 are a crucial tool to produce uncountable families of mutually separated n-dimensional operator spaces for each $n > 2$. The latter families are constructed by a direct sum argument. Since each such family is non-separable (and consists of spaces at rather large distance to each other), we are sure that (uncountably) many of the spaces in that family are not 1-exact (and even have rather large exactness constants). Actually we will describe below a more explicit subclass formed of non-exact spaces. But it turns out that the precise estimate in Theorem 3.12 allows us to specify the construction of these direct sums so that all of them are subexponential. In fact we can construct them as n-dimensional spaces such that their associated sequence $K_E(N,C)$ is of order N^2 when $N \to \infty$. Thus they are all subexponential.

Consider an unbounded increasing sequence of integers $N(0) < N(1) < \cdots < N(m) < \cdots$.

Our example will be of the form

$$E = \mathrm{span}\{x_1, \cdots, x_n\} \quad \text{where} \quad x_j = \oplus_{m\in\mathbb{N}} x_j(m) \quad \text{and} \quad x_j(m) \in M_{N(m)}.$$

For any subset $A \subset \mathbb{N}$, we will denote

$$x_j(A) = \oplus_{m\in A} x_j(m) \quad \text{and} \quad E_A = \mathrm{span}\{x_1(A), \cdots, x_n(A)\}. \tag{6.1}$$

The case $m = 0$ in special. We set $N(0) = 2n$ and

$$x_j(0) = e_{1j} \oplus e_{j1} \in M_n \oplus M_n \subset M_{2n}. \tag{6.2}$$

Thus the space $E_{\{0\}}$ coincides with the space that is usually denoted by $R_n \cap C_n$: for any k and any $a \in M_k^n$ we have $\|\sum x_j(0) \otimes a_j\| = \|a\|_{RC}$. (Actually the latter space $R_n \cap C_n$ embeds in $M_n \oplus \mathbb{C} \subset M_{n+1}$).

We will often refer to the "natural" mapping $E_B \to E_A$ between two such spaces. By this we mean the mapping (induced by the identity on $\mathbb{C}^n$) taking $x_j(B)$ to $x_j(A)$.

Lemma 6.1. *Fix $C > 1$ and m such that $n \leq N(m)$. Assume that for any any $k \leq N(m)$ (or simply for $k = N(m)$) we have for any $(y_j) \in M_k^n$*

$$\|y\|_{RC} \leq \sup_{m'>m} \|\sum x_j(m') \otimes y_j\| \leq C\|y\|_{RC}.$$

Then for the associated space E we have $k_E(N(m), C) \leq m + 1$ and

$$K_E(N(m), C) \leq \sum\nolimits_{0\leq m'\leq m} N(m')$$

Proof. Consider the embedding $J: \ E \to \oplus_{0<m'\leq m} M_{N(m')} \oplus M_n \oplus M_n$ defined by

$$J(x_j) = x_j([0, m]),$$

let F be the image of J and let $u: \ E \to F$ be the same as J but viewed into F. Then (recalling (6.2)) our assumption implies $\|u\|_{cb} \leq 1$ and $\|u^{-1}\|_{cb} \leq C$. □

Outline: Our examples will be obtained by first choosing the sequence $N(m)$ and then setting $x_j(m) = Y_j^{(N(m))}(\omega)$ with ω in a set of positive probability.

The following well known fact will be convenient. It is an easy consequence of the weak convergence (i.e. in moments) when $m \to \infty$ of $(Y_j^{(N(m))})$ to a circular sequence.

Lemma 6.2. *Let $N(0) \leq N(1) \leq \cdots \leq N(m) \leq \cdots$ be any unbounded sequence of integers. For almost all ω in our probability space Ω, the following property holds: for any $k \geq 1$ and any $(y_j) \in M_k^n$ we have*

$$\|y\|_{RC} \leq \liminf_{m\to\infty} \|\sum Y_j^{(N(m))}(\omega) \otimes y_j\|.$$

Proof. This a.s. property is well known for given fixed $k \geq 1$ and $(y_j) \in M_k^n$. Therefore, it also holds for all k and all y's with rational entries. As is well known we have $\sup_N \|Y_j^{N)}\| < \infty$ almost surely for each j, therefore the function $y \mapsto \|\sum Y_j^{(N(m))}(\omega) \otimes y_j\|$ is uniformly continuous for almost all ω. Then the conclusion follows by density. □

We will use Theorem 3.12. Fix $0 < \varepsilon < 1$. Let us choose $N(m)$ inductively such that $N(0) = 2n$ and for any $m \geq 0$

$$\forall m \quad N(m+1) \in [c_\varepsilon n N(m)^2, c_\varepsilon n N(m)^2 + 1]. \tag{6.3}$$

We may assume $c_\varepsilon \geq 1$ so that, since $N(m+1) \geq N(m)^2$ for all m and $N(0) \geq 2$, we must have

$$\forall m \geq 0 \quad N(m) \geq 2^{2^m}. \tag{6.4}$$

Then Theorem 3.12 tells us that the event

$$\Omega_m = \{\forall y \in M_{N(m)}^n \quad \|\sum y_j \otimes Y_j^{(N(m+1))}(\omega)\| \leq (2+\varepsilon)\|y\|_{RC}\}$$

occurs with probability $\mathbb{P}(\Omega_m) \geq 1 - \varepsilon_m$ with $\varepsilon_m = (3^{nN(m)^2} N(m) N(m+1))^{-1} \leq 3^{-2^{2m}}$. Since $\sum \varepsilon_m < \infty$, by Borel-Cantelli the set $\liminf \Omega_m = \cup_{m_0} \cap_{m \geq m_0}$ satisfies

$$\mathbb{P}\{\liminf \Omega_m\} = 1.$$

Let $\Omega' \subset \Omega$ be the event described in Lemma 6.2.
Let $\Omega'' \subset \Omega$ be the set of all ω's such that

$$\limsup_{m\to\infty} \|(Y_j^{(N(m))}(\omega))\|_{RC} \le 2\sqrt{n}.$$

Clearly $\mathbb{P}(\Omega'') = 1$ (Indeed this follows a fortiori from Theorem 3.12).
Let $\Omega''' \subset \Omega$ be the set of all ω's such that for any matrix $a \in M_n$ we have

$$|\mathrm{tr}(a)| \le \liminf_{m\to\infty} |\sum\nolimits_{ij} a_{ij} N(m)^{-1} \mathrm{tr}(Y_i^{(N(m))} Y_j^{(N(m))*})|.$$

The convergence in moments (to a circular family) of $Y_j^{(N(m))}$ when $m \to \infty$ ensures that this event has full probability for any fixed a, but again a density argument (in M_n) ensures that $\mathbb{P}(\Omega''') = 1$.

We now choose ω in the set $\Omega' \cap \Omega'' \cap \Omega''' \cap \liminf \Omega_m$ which occurs with full probability and we set

$$x_j(m) = Y_j^{(N(m))}(\omega) \in M_{N(m)}.$$

By the choice of ω we know that for some m_0 we have for any $m \ge m_0$ and any $(y_j) \in M_{N(m)}^n$

$$\|y\|_{RC} \le \sup_{m'>m} \|\sum x_j(m') \otimes y_j\| \le (2+\varepsilon)\|y\|_{RC} = (2+\varepsilon)\|\sum x_j(0) \otimes y_j\| \tag{6.5}$$

so that

$$\sup_{0\le m'\le m} \|\sum x_j(m') \otimes y_j\| \le \|\sum x_j \otimes y_j\| \le (2+\varepsilon) \sup_{0\le m'\le m} \|\sum x_j(m') \otimes y_j\| \tag{6.6}$$

and hence by Lemma 6.1 and by (6.3) we must have for any $m \ge m_0$

$$K_E(N(m), 2+\varepsilon) \le \sum\nolimits_{0\le m'\le m} N(m') \le 2n + \sum\nolimits_{1\le m'\le m} N(m'),$$

and since it is easy to check that there is a constant γ such that $\sum_{1\le m'\le m} N(m') \le \gamma N(m)$, and since $N(m) \le c_\varepsilon n N(m-1)^2 + 1$ we find

$$K_E(N(m), 2+\varepsilon) \le 2n + \gamma(c_\varepsilon n N(m-1)^2 + 1).$$

Therefore, whenever $N(m-1) < N \le N(m)$ we have $N(m-1)^2 \le N^2$ and hence

$$K_E(N, 2+\varepsilon) \le K_E(N(m), 2+\varepsilon) \le 2n + \gamma(c_\varepsilon n N^2 + 1) \in O(N^2).$$

Note also that

$$k_E(N(m), 2+\varepsilon) \le m + 2.$$

Thus we have proved:

Lemma 6.3. *With the above choice of* $\{N(m)\}$ *and* $x_j(m)$ *the space* E *is subexponential with constant* $C(E) \le 2+\varepsilon$. *Moreover, the same is true for any space* E_A, *as defined in* (6.1), *for any infinite subset* $A \subset \mathbb{N}$.

Proof. The only property we used to check this for the space E, namely (6.5), remains valid for E_A. Indeed, if we restrict to $m' \in A$ the upper bound in (6.5) remains trivially valid, while the lower bound is guaranteed by Lemma 6.2 since $\omega \in \Omega'$. □

Remark 6.4. By technical refinements, it is possible to replace $2+\varepsilon$ by 1 in the preceding statement. To do this one should replace the space $R_n \cap C_n$ by the span of an n-tuple of free circular elements (C_j) in the sense of [40], that is known to be 1-exact. The norm $\|y\|_{RC}$ should then replaced by $\|\sum C_j \otimes y_j\|$, and the results of [11] should be replaced by those of [12]. Moreover, one can actually exhibit a 1-subexponential C^*-algebra which is not exact. See our more recent paper on this on arxiv. In another direction, it should be possible to use Collins and Male's results [4] to replace Gaussian random matrices by unitary ones (uniformly distributed according to Haar measure), but we did not fully check this. For this last point the Gromov-Lévy isoperimetric inequality (see [10, §1.2 and §3.4]) should be used on $U(N)^n$ instead of the Gaussian concentration of measure.

The following was used in [16] for diagonal matrices, the general case was observed in [23].

Lemma 6.5. *With the above notation, let $A, B \subset [m_0, \infty)$ be infinite such that $A \cap [N, \infty) \not\subset B \cap [N, \infty)$ for any $N \in \mathbb{N}$. For any $[a_{ij}] \in M_n$ let $u:\ E_B \to E_A$ be defined by $u(x_j(B)) = \sum_i a_{ij} x_i(A)$. Then we have*

$$(2+\varepsilon)^{-2}(\sum |a_{ij}|^2)^{1/2} \le \|u\|_{CB(E_B,E_A)} \le (2+\varepsilon)(\sum |a_{ij}|^2)^{1/2}.$$

Proof. Note that $\|\sum a_{ij} x_i(0) \otimes x_j(0)\| = (\sum |a_{ij}|^2)^{1/2}$.
Since we choose $\omega \in \Omega'$, Lemma 6.2 ensures that we have a completely contractive natural map $E_B \to R_n$ (and also $E_B \to C_n$) for any infinite B. Therefore $\|u\|_{CB(E_B,E_A)} \le \|u\|_{CB(R_n,E_A)} = \|\sum a_{ij} e_{i1} \otimes x_j(A)\| \le \|\sum a_{ij} x_i(0) \otimes x_j(A)\|$ and hence by (6.5)

$$\|u\|_{CB(E_B,E_A)} \le (2+\varepsilon)\|\sum a_{ij} x_i(0) \otimes x_j(0)\| = (2+\varepsilon)(\sum |a_{ij}|^2)^{1/2}.$$

To prove the converse, we will choose m_0' arbitrarily large in $A - B$. Then (6.5) shows that

$$\|\sum b_{jk} x_j(B \cap (m_0', \infty)) \otimes x_k(m_0')\| \le (2+\varepsilon)\|\sum b_{jk} x_j(0) \otimes x_k(m_0')\| \le (2+\varepsilon)^2 \|\sum b_{jk} x_j(0) \otimes x_k(0)\|,$$

and also since $m_0' \notin B$

$$\|\sum b_{jk} x_j(B \cap [m_0, m_0']) \otimes x_k(m_0')\| \le (2+\varepsilon)\|\sum b_{jk} x_j(B \cap [m_0, m_0']) \otimes x_k(0)\|$$
$$\le (2+\varepsilon)^2 \|\sum b_{jk} x_j(0) \otimes x_k(0)\|,$$

where for the last inequality we use the fact that $N(0) = 2n \le N(m-1)$ for all $m \ge 1$ and in particular for all $m \in B \cap [m_0, m_0']$. Recollecting the two preceding estimates, we find

$$\|\sum b_{jk} x_j(B) \otimes x_k(m_0')\| \le (2+\varepsilon)^2 \|\sum b_{jk} x_j(0) \otimes x_k(0)\|.$$

Thus we have

$$\|\sum_{ijk} b_{jk} a_{ij} x_i(A) \otimes x_k(m_0')\| = \|\sum b_{jk} u(x_j(B)) \otimes x_k(m_0')\| \le \|u\|_{cb}(2+\varepsilon)^2 \|\sum b_{jk} x_j(0) \otimes x_k(0)\|.$$

Since $m_0' \in A$, a fortiori

$$\|\sum_{ijk} b_{jk} a_{ij} x_i(m_0') \otimes x_k(m_0')\| \le \|u\|_{cb}(2+\varepsilon)^2 \|\sum b_{ij} x_i(0) \otimes x_j(0)\|.$$

But now, since $\omega \in \Omega'''$, for any $\delta > 0$, when m_0' is chosen large enough, we have

$$|\sum_{ij} a_{ij} b_{ji}| \le \|\sum_{ijk} b_{jk} a_{ij} x_i(m_0') \otimes x_k(m_0')\| + \delta$$

and hence choosing simply $b_{ij} = \bar{a}_{ji}$ (and letting $\delta \to 0$) we conclude

$$\sum_{ij} |a_{ij}|^2 \leq \|u\|_{cb}(2+\varepsilon)^2 \|\sum \bar{a}_{ji} x_i(0) \otimes x_j(0)\| = \|u\|_{cb}(2+\varepsilon)^2 (\sum_{ij} |a_{ij}|^2)^{1/2}$$

and the announced result follows after division. □

Lemma 6.6. *Fix $m \in \mathbb{N}$. Assume that A, B are infinite subsets such that $A \cap [0,m] = B \cap [0,m]$. Then the mapping $u: E_B \to E_A$ (induced by the identity of $\mathbb{C}^n$) is such that*

$$\forall k \leq N(m) \quad \|u_k : M_k(E_B) \to M_k(E_A)\| \leq 2 + \varepsilon$$

Proof. For all $y \in M_k^n$ we have

$$\|\sum x_j(A \cap [0,m]) \otimes y_j\| = \|\sum x_j(B \cap [0,m]) \otimes y_j\| \leq \|\sum x_j(B) \otimes y_j\|.$$

But also by (6.5) since $k \leq N(m)$ and $\omega \in \Omega'$

$$\|\sum x_j(A \cap [m+1,\infty)) \otimes y_j\| \leq (2+\varepsilon)\|y\|_{RC} \leq (2+\varepsilon)\|\sum x_j(B) \otimes y_j\|.$$

Therefore $\|u_k\| \leq 2 + \varepsilon$. □

The method of the paper [16] as presented in [30, Th. 21.13, p. 343] shows that there is an continuous subcollection in the family $\{E_A \mid A \subset \mathbb{N}, |A| = \infty\}$ such that $ex(E) \geq \sqrt{n}/(2+\varepsilon)^3$. But actually we can make this slighly more precise:

Theorem 6.7. *For any infinite subset $A \subset \mathbb{N}$ we have*

$$ex(E_A) \geq \sqrt{n}/(2+\varepsilon)^3.$$

Proof. Fix A as in the statement. For any $m \geq 0$ let

$$\mathcal{C}_m = \{B \subset \mathbb{N} \mid |B| = \infty,\ A \cap [0,m] = B \cap [0,m],\ A \cap [N,\infty) \not\subset B \cap [N,\infty)\ \forall N \in \mathbb{N}\}.$$

Clearly this set is non empty. For each m, pick $B(m) \in \mathcal{C}_m$. Let $u(m) : E_{B(m)} \to E_A$ denote the natural (identity) map. By Lemma 6.5 we have $\|u(m)\|_{cb} \geq \sqrt{n}/(2+\varepsilon)^2$. However, by Lemma 6.6, for any $k \leq N(m)$ we have $\|u(m)_k\| \leq (2+\varepsilon)$. Assume now that E_A is exact with constant $< c$, so that for some finite k there is $F \subset M_k$ with $d_{cb}(E_A, F) < c$. Choosing m large enough we may ensure that $k \leq N(m)$. By the Smith Lemma 2.2, this implies that $\|u(m)\|_{cb} \leq c\|u(m)_k\|$. Thus we obtain $\sqrt{n}/(2+\varepsilon)^2 \leq c(2+\varepsilon)$, and hence $c \geq \sqrt{n}/(2+\varepsilon)^3$. □

Recapitulating, we can now conclude

Theorem 6.8. *For any $n \geq 1$ and $\varepsilon > 0$, there is a continuum of n-dimensional, subexponential (with constant $2+\varepsilon$) operator spaces with mutual cb-distance $\geq n/(2+\varepsilon)^4$ and with exactness constant $\geq \sqrt{n}/(2+\varepsilon)^3$.*

Note that the preceding lower bounds are not significant for small values of n.

Remark 6.9. Note that the preceding result can also be obtained using the main idea of [31]. In fact that idea proves more generally that for A infinite with infinitely many gaps the space E_A has a large d_f constant of embedding into $C^*(\mathbb{F}_2)$ in the sense of [30, p. 345].

Corollary 6.10. *There is a separable infinite dimensional subexponential operator space that is not exact.*

Proof. Let $E(n)$ be an n-dimensional example as in the Theorem. Let X be the direct sum in the c_0-sense of $\{E(n)\}$, so that the elements of X are sequences $x = x(n)$ tending to zero in norm and $X \subset \oplus_n E(n)$. We claim that any finite dimensional subspace $E \subset X$ is subexponential with constant $2 + \varepsilon$. By perturbation, it suffices to show this for $E = \oplus_{[0 \le n \le q]} E(n)$, for any integer q. But now an easy verification shows that for such an E we have

$$K_E(N, C) \le \sum_{0 \le n \le q} K_{E(n)}(N, C)$$

and since each $E(n)$ is subexponential with constant $2 + \varepsilon$, we conclude that E also is. □

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